Voltage Feasibility-Constrained Peer-to-Peer Energy Trading with Polytopic Injection Domains

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Abstract—Peer-to-peer (P2P) energy trading is an important energy market concept that improves the utilization of distributed energy resources and promotes the integration of energy storage technologies in distribution grids. It is challenging to satisfy the grid operational feasibility under such decentralized energy markets while enabling fully autonomous prosumer operations. This work develops a self-validation mechanism based on polytopic injection domains that define the allowed/safe region of prosumer power injections, ensuring feasible operation of the distribution grid. The allowed polytopic injection domains are constructed effectively by leveraging a newly developed feasibility criterion based on Kantorovich’s fixed-point theorem. Therewith, we design a novel P2P energy market framework supporting the autonomous participation of prosumers and propose a nodal aggregator model to validate the aggregated prosumer power injections to attest a voltage-feasible market-clearing.

Index Terms—Active distribution grids, fixed-point theorem, peer-to-peer energy trading, voltage feasibility.

I. INTRODUCTION

Distributed energy trading has been a recent trend in distribution grids due to the increased penetration of distributed energy resources (DERs) such as renewables, energy storage, and flexible loads, and the advancements in communication technologies [1], [2]. Peer-to-peer (P2P) energy market frameworks allow prosumers to establish contractual agreements for bilateral energy-sharing between themselves [3]. Here, prosumers are the entities who own several DERs, and having the capability to produce/consume in a particular market interval depending on their preferences [4]. There are different levels of P2P energy markets scaling from community-level to feeder-level to grid-level applications in the distribution grids. In the latter case, P2P energy markets function alongside the existing retail power market arrangements. As prosumers can be located in different nodes/buses of the distribution grid, these P2P prosumer energy transactions utilize the distribution network, and hence, cause changes in its power flow [5], [6]. Thus, P2P energy trading contracts must be established considering the nodal power injection amounts that can be absorbed by the distribution grid without causing grid constraint violations.

The frameworks proposed in the literature to address this issue consider these two aspects. Firstly, a centralized-entity is considered as an active market player who negotiates with prosumers for the cost of dispatching additional ancillary services to satisfy the grid constraints [5]. For instance, Ref. [7] consider prosumers at both transmission and distribution grid levels, and negotiate with both the transmission and distribution system operators (TSO and DSO) for P2P energy market clearing. Refs. [8], [9] compute distribution locational marginal price-based grid utilization fees to share the costs/;rewards of ancillary services associated with P2P energy transactions. Ref. [10] extends it to a more accurate form with an iterative algorithm considering the negotiations with the DSO for dynamic changes of distribution locational marginal prices. Such a third-party validation reduces the autonomy of the P2P energy markets as the centralized entity, such as both the TSO and DSO in [7], the DSO in [10], or the coordinator in [5], is designed such that it knows all the prosumer locations and the corresponding power injections in each iteration within the market-clearing process. Thus, the centralized entity can directly influence the P2P energy market equilibrium. Further, solving optimal power flow in each iteration as in [10] significantly increase the computational burden of the P2P energy market clearing mechanism. Secondly, the grid constraints are approximated with linear functional forms in terms of prosumer power injections. Therein, Ref. [7] employs linearized power flows, Ref. [11] employs first-order approximations for power losses, and Ref. [12] employs voltage, power transfer, and loss distribution factors via first-order sensitivities. Importantly, the aforementioned linearized formulations or first-order sensitivities does not guarantee the accuracy and often fails outside their neighbourhood region leading to invalid or infeasible P2P energy transactions.

The regions that a feasible equilibrium exists quantify the safe power-injection variation domains of the aggregated prosumers without harming the stability of the distribution system. Estimating the region associated with the existence of steady-state equilibrium and the operational constraints are essentially difficult due to the nonlinearity and the high dimensions of power systems. The approaches that attempt to construct such
regions [13], including the continuation power flow (CPF) [14], holomorphic embedding load-flow method (HELM) [15], optimization [16] and machine learning based techniques [17], suffer from one or many of the following difficulties: non-convexity, scenario-based, sensitive to the initial points or step size, relaxation gaps, additional simplifications/assumptions, etc., [16], [18]. Thus, there is a lack of a systematic approach which is handy and robust to construct the convex feasibility regions under complex scenarios. Hence, this work leverages the feasibility certificate developed based on Kantorovich fixed-point theorem [19], which has norm-based forms and applicable to general power systems under various working conditions. The norm-based form offers convex polytopic domains that saves extra treatment for convexity and contributes to computational efficiency. The convergence of Newton-Raphson iterations is promised by Kantorovich fixed-point theorem, which guarantees the robustness of its performance under complex P2P scenarios and network parameters. With this method, DSO can determine the polytopic sets at each bus of the system, and the P2P energy transactions which are self-validated with respect to these polytopic sets can satisfy the grid feasibility requirements automatically.

The contributions in this work are as follows.

1) We design a novel P2P energy market framework supporting the autonomous participation of prosumers. The role of DSO in the proposed P2P energy market is providing the aforementioned feasible injection domains for prosumer nodes prior to the initiation of the P2P market clearing mechanism.

2) We propose a third-party-free self-validation mechanism that relies on the concept of the feasible injection domains, validating safe nodal injection amounts. Therewith, the proposed P2P energy market framework guarantees grid feasibility without iteratively solving the non-linear power flow equations or considering the DSO as an active market participant, i.e., prosumers do not share their power injections with the DSO within the market mechanism.

3) We further propose a nodal aggregator (NA) model to validate the aggregated prosumer power injections to attest a voltage-feasible market clearing. Utilization of convex polytopic injection domains ensures the voltage feasibility of the resulting P2P energy trading contracts. Further, it aids in formulating convex optimization models for both prosumers and NAs supporting scalability for larger participation of prosumers.

4) Finally, we prove that the proposed market clearing algorithms converge to the global optimality.

While using the feasible subsets reduces economic benefits, the computation is more efficient due to the convexity of decentralized market clearing algorithms. Here, feasibility subsets are constructed efficiently, for example, by inscribing polytopic sets as in Fig. 1 based on the Kantorovich fixed-point theorem as explained in Section II [20].

\textbf{Notation—}The sets of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C} \) respectively, and the set of nonnegative real numbers by \( \mathbb{R}_+ \). For a matrix \( A \), its transpose is \( A^T \) and its element in row \( i \) and column \( j \) is \( A_{ij} \). \( [x] \) is the diagonal matrix with its diagonal element is \( x \). \( 1 \in \{1\}^n \) with compatible vector size. \( \overline{\mathbf{x}} \) is the conjugate of \( \mathbf{x} \). \( \| \cdot \| = \| \cdot \|_\infty \) unless specified otherwise.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{Fig1.png}
\caption{A polytopic subset of the feasibility set.}
\end{figure}

\section*{II. Feasibility Set Construction using the Kantorovich Fixed-Point Theorem}

In this work, the feasibility set is constructed by a newly derived feasibility certificate [19] which is based on the Kantorovich fixed-point theorem. The constructed feasibility set quantifies the region where an equilibrium exists in the distribution system, such that the voltage stability and operational constraints can be guaranteed as long as the variations of power injections maintains inside the set. The basics of the Kantorovich fixed-point theorem are as below.

\textbf{Theorem 1} (Kantorovich fixed-point theorem), Let \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be an analytic vector-valued complex function. Let \( J(x) = \frac{\partial F}{\partial x} \) be the Jacobian matrix. Suppose that \( \exists \Omega \subseteq \mathbb{C}^n, x_0 \in \Omega \) such that the following conditions are satisfied:

\begin{align}
\det (J(x_0)) & \neq 0, \quad (1a) \\
\left\| (J(x_0))^{-1} (J(x) - J(y)) \right\| & \leq L \|x - y\| \quad \forall x, y \in \Omega \quad (1b) \\
\left\| (J(x_0))^{-1} (F(x_0) - b) \right\| & = \eta \leq 1/(2L) \quad (1c)
\end{align}

Then, \( F(x) = b \) has a unique solution in the set \( \mathcal{B} = \{ x : \|x - x_0\| \leq (1 - \sqrt{1 - 2L\eta})/L \} \subseteq \Omega \) and Newton’s method initialized at \( x_0 \) converges to this solution.

This theorem guarantees when the conditions 1a, 1b, 1c satisfy, there is a fixed-point existed inside the ball \( \mathcal{B} \) in Theorem 1. In the context of the power system, this fixed-point indicate the solution of power flow problem. In another words, the Newton-Raphson iterations will converge to an equilibrium inside the ball \( \mathcal{B} \). In order to apply the Kantorovich fixed-point theorem to the power flow equations with a particular choice of nominal point \( x_0 \), as the conjugate operation in \( s = V^T \) makes the function not analytic in the complex variables, we rewrite the power flow equations and the voltage constraints \( V_{\text{min}} \leq |V| \leq V_{\text{max}} \) in the following form:

\begin{align}
[\overline{\mathbf{\theta}}](\zeta(s)\mathbf{\theta} + u) &= \mathbf{1}, \\
\theta \overline{\mathbf{\theta}} - t_1 t_1 & = 1/V^2_{\text{max}}, \quad \theta \overline{\mathbf{\theta}} + t_2 t_2 = 1/V^2_{\text{min}}, \quad (2)
\end{align}
where $\zeta(s) = [V_{\text{nom}}]^{-1}Z[V_{\text{nom}}]^{-1}[s]$, $\theta = [V_{\text{nom}}]_\gamma, u = V^0/V_{\text{nom}}$. $V_{\text{nom}}, s_{\text{nom}}$ are the nodal voltage and power injection of the base operating point. $Z$ is the impedance matrix of the system. $V^0$ denotes the voltage at zero injection point. $\gamma_i = 1/V_i$. $I_i = \gamma_i s_i$. $V_i$ denotes the complex voltage at bus $i$ and $I_i$ is the complex current. $s_i$ denotes the complex power injection at bus $i$. $t_1, t_2 \geq 0$ are two slack variables. The following affine function is introduced for an analytic form of equations where both the structure and the set of solutions of the power flow equations can be preserved.

$$
\hat{F} : C^{2n} \mapsto C^{2n} : \hat{F} \left( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) = \begin{bmatrix} \hat{F}(\beta, \alpha) \\ F(\alpha, \beta) \end{bmatrix}
$$

so that for each solution $\alpha$ of $F(\alpha, \beta) = b$, there exists:

$$
\hat{F} \left( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) = \begin{bmatrix} b \end{bmatrix}
$$

Thus, a new extended set of equations $F(\alpha, \beta) = b, \alpha = [\theta; \beta; t_1; t_1; t_2; t_2], b = [1; V_{\text{nom}}/V_{\text{nom}}]_\gamma$ can be established based on the original power flow equations and voltage constraints. Then the Kantorovich fixed-point theorem can be applied to this extended set to get the feasibility set as below:

$$
\|M \Delta s\| \leq 1/(4\kappa)
$$

where

$$
M = [V_{\text{nom}}]^{-1}Z[V_{\text{nom}}]^{-1}
$$

$$
\kappa = \|J_{\text{nom}}^{-1}\| \left( \|s_{\text{nom}}\| \|J_{\text{nom}}^{-1}\| + 1 \right)
$$

Note that the subscript $\text{nom}$ refers to the nominal point, thus the $J_{\text{nom}}, \zeta_{\text{nom}}, V_{\text{nom}}$ that refer to the nominal point are pre-calculated values. $\Delta s = s - s_{\text{nom}}$ represents the loading variation. When only active power changes are considered in the case of P2P trading, $\Delta s = \Delta P = P - P_{\text{nom}}$ and (5) can be reduced to the following inequality with $r = 1/(4\kappa)$:

$$
\|M \Delta P\| \leq r
$$

which can be rewritten as:

$$
\sum_j |M_{ij}||\Delta P_j| \leq r \quad \forall i
$$

This feasibility set converts the nonlinear power flow constraints to a convex body that offers great computational benefits supporting the self-validation of nodal power injections in P2P transaction.

Remark 1. In general, the proposed Kantorovich fixed-point theorem-based approach has the advantages on the computational cost, no limitation on network topology, the robustness to system operating points and parameters, and no simplification on system equations and constraints. Whereas, most of the optimization-based approaches may incur inherent difficulties such as convex relaxation, initialization, high computational cost, simplification of the constraints, etc., [21].

To this end, the proposed approach enables grid feasibility via the set of constraints (7) constituting the parameters $M$ and $r$ that must be jointly satisfied by the overall prosumer power injections. Such an approach is less conservative compared to the optimization-based region construction approaches [22]–[24] that provides marginalized nodal injection limits.

Another point to note is that sets of feasible operation points must not meet singular boundaries. Recently there have been major efforts developing computational differential geometric methods estimating the distance to the singular boundary in voltage and power space [25] while assuring that the shortest path to the singular boundary is partial to a connected submanifold of admissible states defined by nonlinear physical constraints [25]–[27]. Ref. [26] computes shortest paths in a constraint manifold joining operation points with the voltage instability boundary while observing the influence of frequently changing highly dispersed distributed energy resources. Ref. [25] computes delicate curvature structures of singular boundary manifolds and discusses methods for computing geodesics in submanifolds defined by constraints including submanifolds of singular boundaries. Using curvature properties of the singular boundary in power space [28] early and nicely presented methods for computing nearest points on the singular boundary in power space. For slow/fast vector fields of algebraic ordinary differential equation (ODE) systems [29] using geodesic coordinates finds and traces curves of points in a jump set component of an algebraic manifold component together with respective points on a hit set in the same manifold component (ODE solutions meeting the jump set change speed and guided by the fast vector field hit points in the related hit set on some of the many manifold components. Here, it is difficult to decide if a component is containing the jump set. This is however assured by [29]).

III. VOLTAGE-FEASIBLE P2P ENERGY MARKET DESIGN

In this section, we present the proposed market design to facilitate P2P energy trading in distribution grids while preserving privacy and autonomy of market participants. Prosumer optimization models are constructed for determining the optimal energy trading quantities with other prosumers participated in the P2P energy market, while respecting its own capability limits. As such, prosumers form bilateral energy sharing contracts with other prosumers to maximize their welfare and local asset utilization. A bilateral contract includes the energy trading amount and the per unit price that will be negotiated within the P2P market mechanism. Here, prosumers can participate in negotiations using anonymous IDs for different market intervals so that their identity cannot be traced and the energy trading preferences cannot be predicted in future.

As these prosumers are typically located at different nodes/buses over the distribution grid, the energy transactions among themselves will be transferred over the distribution network. Therefore, these energy transactions can violate the safe operational limits of the distribution grid [30]. In this work, we utilize the Kantorovich fixed-point theorem to construct safe power injection region for each node of the distribution grid. Restricting the aggregated prosumer power injections at every node within the corresponding polytopic region (7) ensures the
The feasibility of the grid constraints as proven in Section II. In the proposed P2P energy market framework, the DSO clears the retail energy market and determines the polytopic sets for each prosumer node/bus of the distribution grid. These polytopic sets are broadcasted throughout the entire network prior to the P2P energy market clearing process. Therewith, we establish NAs for each prosumer node and assigns them to validate the aggregated prosumer power injections within the prespecified polytopic injection regions (7). Here, the prosumers only share their total power injection/extraction amounts only with the NA corresponding to their operating node. Therefore, bilateral energy trading contract information with other prosumers is still preserved within each prosumer. Further, the prosumers can use a separate anonymous ID (different to the one they use for negotiating with other prosumers) to communicate with the corresponding NA within the market clearing mechanism. Fig. 2 provides an overview of the proposed P2P energy market framework. The distributed market clearing is facilitated via alternating direction method of multipliers (ADMM). We establish prosumer and NA optimization models and the price update mechanism following the standard structure of ADMM. Therein, each prosumer and NAs iteratively execute a predefined optimization problem, share some boundary variables (information relevant to both parties), and update respective bids to reach consensus among them under a predefined tolerance. These optimization models and the market clearing algorithms are presented.

A. Prosumer Optimization Model

In this work, we assume the prosumer optimization with a generic model considering dispatchable capability limits. Let \( P \) be the set of prosumers in the P2P energy market. Let, the vector \( e^p_i \in \mathbb{R}^{|P|} \) represents the total energy production vector of prosumers, in which a positive value implies production and a negative value implies consumption. Let the vector \( \Lambda^p_i \in \mathbb{R}^{|P|} \) represents the respective bid prices which will be iteratively determined in the proposed market clearing mechanism. Importantly, that reflects the share of prosumer \( i \) to cover the cost of losses increased due to P2P energy transactions, cost of network utilization, and service fee for providing the valid nodal injection regions. Let the matrix \( E = [E_1; E_2; \ldots; E_i; \ldots; E_{|P|}] \in \mathbb{R}^{|P| \times |P|} \) represents the bilateral energy transactions between prosumers. Therein, \( E_i \in \mathbb{R}^{1 \times |P|} \) is the vector comprising the energy trading quantities of prosumer \( i \) preferred to be traded with other prosumers, and \( E_{i,j} \) is the energy sold (positive values) or purchased (negative values) by prosumer \( i \) to or from prosumer \( j \) respectively with \( E_{i,i} = 0 \). Hence, for the compliance of energy transactions between P2P prosumers requires \( E_{i,j} = -E_{j,i} \) and \( E + E^T = 0 \). Let the matrix \( \Pi = [\Pi_1; \Pi_2; \ldots; \Pi_i; \ldots; \Pi_{|P|}] \in \mathbb{R}^{|P| \times |P|} \) represents the energy trading prices between the prosumers in the P2P energy market. Therein, \( \Pi_i \in \mathbb{R}^{1 \times |P|} \) is the vector comprising the energy trading prices of prosumer \( i \) preferred to be traded with other prosumers, and \( \Pi_{i,j} \) is the price at which the energy is sold or purchased by prosumer \( i \) to or from prosumer \( j \) respectively with \( \Pi_{i,i} = 0 \).

To this end, the optimization problem solved by P2P prosumer \( i \in P \) can be formulated as follows:

\[
\min_{e^p_i, E_i} \mathcal{O}^\text{P2P}_i = \mathcal{H}_i(e^p_i) - \Lambda^p_i e^p_i - \Pi_i E_i^T + \frac{\rho}{2} \| E_i - \hat{E}_i \|_2^2
\]

\[
+ \frac{\rho}{2} (e^p_i - \bar{e}^p_i)^2
\]

s.t. \( e^p_i - E_i |_{|Z_i|} = 0 \), \( E_{i,i} = 0 \)

\[
\epsilon^p_i \leq e^p_i \leq \epsilon^p_i \]

where \( \mathcal{H}_i : \mathbb{R} \rightarrow \mathbb{R} \) is the convex cost function of prosumer \( i \in P \). The general shapes of the objective functions are explained in [4, Sec. II-A]. Power balance at each prosumer is satisfied by (8b). The power production/consumption limits are respected by (8c). Moreover, \( \bar{e}^p_i \) and \( \hat{E}_i \) are the predetermined fixed values of the coupling variables \( e^p_i \) and \( E_i \) respectively. The deviations are penalized via 2-norm with the weight parameter \( \rho \) in the objective function following the standard problem formulation of ADMM. These fixed values and the price bids \( \Lambda^p_i \) and \( \Pi_i \) are iteratively updated to achieve consensus between distributed agents in ADMM.

B. Nodal Aggregator Optimization Model

We define NA optimization considering the self-validation of the aggregated prosumer power injections with the aim of satisfying the grid constraints. Typically, there are multiple prosumers at a single bus and some of the buses might be free of prosumers. Let \( M \) be the set of buses in the distribution grid at which the prosumers are physically connected. Accordingly, we establish the set of NAs \( M \) in the proposed P2P energy market framework. Let \( Z : P \rightarrow M \) be an onto function that maps the prosumers to the corresponding NAs of the distribution grid. Then, let \( P(n) = \{ i \in P : Z(i) = n \} \subseteq P \) be the set of P2P prosumers connected to bus \( n \) with \( \Xi_n :
\( \mathcal{P}(n) \rightarrow \mathcal{Z}(n) \) be a one-to-one and onto function that maps them to a unique set of \( \mathcal{Z}(n) \). Further, let \( e^Z_n \in \mathbb{R}^{\mathcal{Z}(n)} \) be the \( n \)th node \( Z \)-variable for \( n \)th P2P active power injection with the respective bid price vectors \( A^Z_n \in \mathbb{R}^{\mathcal{Z}(n)} \). Here, \( e^Z_n \) couples the optimization model of NA with P2P prosumer optimization models.

Let \( p \in \mathbb{R}^{\mathcal{M}} \) be the vector of aggregated nodal prosumer power injections, and that must be subjected to (7) in order to satisfy the grid constraints following the self-validation certificate generated in Section II. Note that (7) is a global set of constraints and we must design a method to satisfy them locally to adhere with the distributed nature of the proposed P2P energy market framework. Therein, let \( \{p_1, p_2, \ldots, p_n, \ldots, p_{\mathcal{M}}\} \) be a set of local copies of \( p \) where \( p_n = [p_{n,1}, p_{n,2}, \ldots, p_{n,n}, \ldots, p_{n,\mathcal{M}}]^T \in \mathbb{R}^{\mathcal{M}} \) with \( p_{n,m} \) be the aggregated nodal prosumer power injection at node \( m \) of the distribution grid estimated by NA \( n \). As such, for all \( n \in \mathcal{M} \), NA \( n \) computes \( p_n \) to minimize the grid interaction costs at node \( n \) subject to (7) and other technical constraints. Nodal aggregators negotiate via bid prices \( \lambda_n \in \mathbb{R}^{\mathcal{M}} \) at each iteration to achieve the consensus of \( p^* = p_1 = p_2 = \ldots = p_n = \ldots = p_{\mathcal{M}} \) at the termination.

\[
\min_{\tilde{p}, \tilde{p}, p, e^Z_n} \sum_{n} \tau_n \tilde{p}^2_n + \sum_{n} \tilde{p}^2_n + \lambda^T_n p_n + (A^Z_n)^T e^Z_n + \frac{\rho}{2} \|p_n - \tilde{p}_n\|^2 + \frac{\rho}{2} \|e^Z_n - \tilde{e}^Z_n\|^2 \tag{9a}
\]

s.t. \( p_n \leq \tilde{p}_n, -p_n \leq \tilde{p}_n \),

\( A \tilde{p}_n \leq \lambda_{\mathcal{M}, b} \),

\( p_{n,m} - \tilde{p}_{n,m}^T e^Z_n = 0 \),

\( \tilde{p}_{\mathcal{M}, n} = 0 \) \tag{9e}

where \( \tilde{p} = [p_n] \in \mathbb{R}^{\mathcal{M}} \) is an auxiliary variable introduced to enforce the polytopic injection region (7), i.e., \( A p_n \leq \lambda_{\mathcal{M}, b} \), using a convex reformulation via (9b)-(9c). \( \tau_n p^2_n \) is the share of NA \( n \) to cover the power losses increased in the distribution grid due to the P2P energy transactions, following [4] which models power losses proportional to the square of the nodal power injections. \( \tau_n \tilde{p}^2_n \) is the share of NA \( n \) to cover the network utilization fees and the service fee for providing the allowed nodal power injection regions for self-validation. These amounts are transferred to the prosumers via the dynamic computation of the bid price \( A^Z_n \in \mathbb{R}^{\mathcal{Z}(n)} \) within the P2P energy market clearing mechanism. Constraint (9d) aggregates the power injections from the prosumers located at the operating node with the assumption that NAs do not consume, store, or generate energy. Constraint (9e) locally ensures the power balance within the P2P energy market using the local copy of nodal aggregated P2P power injections \( p_n \). Moreover, \( \tilde{p} \) and \( e^Z_n \) are the predetermined fixed values of the coupling variables \( p_n \) and \( e^Z_n \), respectively.

### C. ADMM-based Coordination and Optimization

The coordination between the coupling variables of (8) (x-update) and (9) (z-update) towards the optimal market clearing is achieved based on the ADMM [31]. Therein, Algorithm 1 and Algorithm 2 are designed for optimization and coordination of prosumer \( i \in \mathcal{P} \) and NA \( n \in \mathcal{M} \) respectively.

#### Algorithm 1: Prosumer \( i \)

**Input:** \( A^P_i(0), \Pi^P_i(0), e^P_i, E_i, \varepsilon_{\varepsilon}, \varepsilon_{E}, k = 0 \)

1. while \( |e^P_i - e^P_{i}^{(k)}| \geq \varepsilon_{E} \) & \( |E_i - E_{i}^{(k)}| \geq \varepsilon_{\varepsilon} \) do

2. Execute (8) and obtain \( e^P_i \) and \( E_i \) \( x \)-update.

3. Send \( e^P_i \) to and receive \( e^Z_{n,i} \) from NA \( n \) where \( n = \mathcal{I}(i), z = \Xi_n(i) \).

4. Send \( E_i^{(k)} \) to and receive \( (E_{j,i}^{(k)}) \in P_{\mathcal{M}} \) from the prosumers \( P \setminus \{i\} \).

5. Update the \( e^P_i \) and \( E_i \) as in (10).

6. Update \( A^P_i \) and \( \Pi^P_i \) as in (11).

7. \( k \leftarrow k + 1 \)

end

**Output:** \( A^P_i, \Pi^P_i, e^P_i, E_i \)

As summarized in Algorithm 1, at iteration \( k \), prosumers execute the local energy management problem (8) in Step 2 to obtain \( e^P_i \) and \( E_i \). Let \( E_{j,i}^{(k)} \) be the coupling variable \( E_{i,j} \) computed by prosumer \( i \) for negotiations between prosumers. In Steps 3 and 4, prosumer \( i \) negotiates with the other prosumers \( P \setminus \{i\} \) and the NA \( \mathcal{I}(i) \) assigned to its physically connected grid node. Then, in Step 5, prosumer \( i \) updates the fixed coupling variables \( e^P_i \) and \( E_i \) in (10),

\[
e^P_i = \frac{1}{2} [e^P_i + e^Z_{n,i}]; \forall i \in \mathcal{P}, n = \mathcal{I}(i), z = \Xi_n(i) \tag{10a}
\]

\[
E_i = \frac{1}{2} \left[ E_{j,i}^{(k)} - (E_{j,i}^{(k)})^T \right]; \forall i \in \mathcal{P} \tag{10b}
\]

and update the bid prices \( A^P_i \) and \( \Pi^P_i \) as in (11) in Step 6 respectively.

\[
A^P_i(k + 1) = A^P_i(k) - \rho \left[ e^P_i(k) - \tilde{e}^P_i(k) \right]; \forall i \in \mathcal{P} \tag{11a}
\]

\[
\Pi^P_i(k + 1) = \Pi^P_i(k) - \rho \left[ E_i(k) - \tilde{E}_i(k) \right]; \forall i \in \mathcal{P} \tag{11b}
\]

Algorithm 1 terminates when the metrics defined in Step 1 reach below the tolerance levels following [31].

As summarized in Algorithm 2, at iteration \( k \), NAs execute (9) in Step 2 to obtain the updated \( p_n \) and \( e^Z_n \). Let \( p_{n,m} \) be the coupling variable \( p_{n,m} \) computed by NA \( n \) for negotiations between NAs. In Steps 3 and 4, NA \( n \) negotiates with the other NAs \( \mathcal{M} \setminus \{n\} \) and the prosumers \( \mathcal{Z}(n) \) physically connected to its operating node. Then, in Step 5, NA \( n \) updates the fixed coupling variables \( \tilde{p}_n \) and \( e^Z_n \) in (12).

\[
\tilde{p}_n = \frac{1}{|\mathcal{M}|} \left[ p_{n,n} + \sum_{m \in \mathcal{M} \setminus \{n\}} p_{m,n} \right]; \forall n \in \mathcal{M} \tag{12a}
\]

\[
e^Z_n = \frac{1}{2} \left[ e^Z_n + (e^P_i)^T \right]; \forall n \in \mathcal{M} \tag{12b}
\]

In Step 6, the computed \( \tilde{p}_n \) is shared among the other NAs \( \mathcal{M} \setminus \{n\} \). Finally, in Step 7, the bid prices \( \lambda_n \) and \( A^Z_n \) are updated as in (13).

\[
\lambda_n(k + 1) = \lambda_n(k) + \rho \left[ p_n(k) - \tilde{p}_n(k) \right]; \forall n \in \mathcal{M} \tag{13a}
\]

\[
A^Z_n(k + 1) = A^Z_n(k) + \rho \left[ e^Z_n(k) - \tilde{e}^Z_n(k) \right]; \forall n \in \mathcal{M} \tag{13b}
\]
Algorithm 2: Nodal Aggregator $n$

Input : $\lambda_{n}(0)$, $A_{n}Z(0)$, $\tilde{p}$, $\tilde{e}^{Z}_{i}$, $\tilde{\varepsilon}_{p}$, $\varepsilon_{e}$, $k = 0$

1 while $\|p_{n} - \bar{p}\|_{\infty} \geq \varepsilon_{p}$ and $\|e_{n}^{Z} - e_{n}^{Z} \|_{\infty} \geq \varepsilon_{e}$ do

2 Execute (9) and obtain $p_{n}$ and $e_{n}^{Z}$: $z$-update.

3 Send $p_{m,n}$ to and receive $p_{m,n}$ from each NA $m$ for all $m \in M \setminus \{n\}$.

4 Send $e_{m,n}$ to and receive $(e^{Z}_{i})_{i \in Z(n)}$ from the set of prosumers $Z(n)$.

5 Update $\tilde{p}_{n}$ and $e_{n}^{Z}$ as in (12).

6 Send $p_{n}$ to and receive $p_{m}$ from each NA $m$ for all $m \in M \setminus \{n\}$.

7 Update $\lambda_{n}$ and $A_{n}Z$ as in (13).

8 $k \leftarrow k + 1$

end

Output: $\lambda^{*}_{n}$, $A_{n}^{*}Z$, $p^{*}_{n}$, $e^{Z*}_{n}$

Algorithm 2 terminates when the metrics defined in Step 1 reach below the tolerance levels following [31].

Assumption 1. There exists a strictly feasible solution for the overall optimization problem defined in (14).

\[
\min_{\lambda_{Pa}, \lambda_{NA}} \sum_{i \in P} \Omega^{P2P}_{i} + \sum_{n \in M} \Omega^{NA}_{n}
\]

s. t. (8b)–(8c); $\forall i \in P$, (9b)–(9e); $\forall n \in M$

(14b)

Theorem 2. The proposed ADMM-based P2P market framework defined in Algorithms 1 and 2 converges to the globally optimal solution of (14), under a proper step-size of $\rho$.

Proof. The objective functions $\Omega^{P2P}_{i}$; $\forall i \in P$ and $\Omega^{NA}_{n}$; $\forall n \in M$ are closed, proper, convex, and continuously differentiable real-valued functions. By construction, (14) constitute convex inequality constraints and affine equality constraints. Hence, with Assumption 1, it satisfies the Slater’s condition for constraint qualification discussed in [32, Ch. 5.9]. Thus, (14) constitutes a saddle-point and holds zero duality gap. Therewith, the proposed ADMM-based distributed market framework satisfies the necessary and sufficient conditions stated under Assumptions 1 and 2 in Chapter 3.2 of [31] for convergence and global optimality. The proof based on the general structure of ADMM can be found in [31, Appendix A].

Therefore, the iterates of Algorithms 1 and 2 satisfy residual convergence, objective convergence, and dual variable convergence as $k \to \infty$ [31, Ch. 3.2.1].

IV. SIMULATIONS

The case studies are performed on a modified IEEE 123-bus distribution system [33], [34] which has 56-buses and bus 56 is the root node of the system. Further, voltage bounds are set within [0.9, 1.0] p.u. to illustrate the effectiveness of the proposed market framework. In this paper, the feasibility regions are compared with the real boundaries calculated by CPF. For the P2P energy market implementation, we randomly selected 30 buses in the system to locate 60 prosumers. Therein, 12 buses, i.e., 1, 2, 3, 12, 17, 27, 33, 35, 36, 40, 49, and 55, were considered as producer buses, and each of them constitutes two prosumers with supply capability as in Type 1 and Type 2 in Table I. Similarly, 18 buses, i.e., 6, 9, 11, 23, 25, 28, 29, 30, 37, 41, 42, 43, 46, 48, 50, 51, 53, and 54, were considered as consumer buses, and each of them constitutes two prosumers with demand capability as in Type 3 and Type 4 in Table I. $\tau_{L} = \$2/MW^{2}h$ and $\tau_{I} = \$1/MWh$. ADMM parameters were selected as $\rho = 0.05$; $\varepsilon_{e} = \varepsilon_{p} = 10^{-3}$ kW; and $\varepsilon_{E} = 10^{-4}$ kW. Algorithms 1 and 2 were programmed in MATLAB, and the optimization problems (8) and (9) were solved using GUROBI. The P2P energy market clearing was simulated on a desktop PC with an Intel® Core(TM) i7-8700 six-core CPU processor running at 3.20 GHz with 16 GB of RAM.

<table>
<thead>
<tr>
<th>Capacity (kW)</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($/MWh$)</td>
<td>15</td>
<td>16</td>
<td>31</td>
<td>30</td>
</tr>
</tbody>
</table>

The performance of the feasibility region construction is demonstrated in Figure 3. Nodes 1 and 2 are selected to plot the feasibility regions constructed by the Kantorovich fixed-point theorem-based certificate. Fig. 3 shows the constructed region (in red) is not overly conservative compared with the real boundary (in black). The polytopic injection domain are demonstrated in Figure 3. Nodes 1 and 2 vary from a load to a power source with the changing direction $\Delta P_{1}/\Delta P_{2} = \tan \varphi$, $\varphi = 0 \sim 2\pi$. The remaining nodal injections maintain a constant power factor of $\cos(\phi) = 0.9$ when power injections change. The polytopic injection values for the P2P trading in the following optimization, i.e. the matrix $M$ in inequality (7) (or $A$ in (9e)), and the selected nodes can be found in the link [35]. Moreover, note that the fixed-point theorem-based approach is more computationally efficient compared to the optimization-based approaches. It averages takes 0.1460 s - 0.376 s to construct a feasible region, whereas, the convex
inner approximation-based method in [22] takes around 10 s with a similar conservativeness on the area of the region.

A. Feasibility Comparison

To show the impact of P2P transactions on the voltage profile, simulations were conducted under two cases.

- Case 1: P2P trading without considering grid constraints.
- Case 2: The proposed market framework (Algorithms 1 and 2 are executed) with self-validation of nodal aggregated prosumer injections.

Fig. 4 shows the voltage profiles under the nodal aggregated prosumer power injections computed in Case 1 and Case 2.

Fig. 5 illustrates the nodal aggregated prosumer power injections at each prosumer bus in the distribution network.

The color-map in Fig. 6 illustrates the P2P transaction amounts $E^p$ in Case 2. The skew-symmetric color distribution reflects the power balance achieved via the P2P market-clearing mechanism. Fig. 7 shows the $\infty$-norms of the residuals evaluated in Algorithms 1 and 2 over the iterations.

Fig. 8 shows the computed nodal aggregated prosumer injections at each prosumer bus in Case 2. The algorithms converge in 198 iterations satisfying the acceptable termination criteria [31]. Excluding the communication overhead, the overall computation time taken is estimated as $T_{total} = \sum_k \max_{i \in P} T_{P2P}^{i,k} + \max_{n \in M} T_{NA}^{n,k}[\max] = 2.59$ seconds, as both the prosumers and NAs operate in parallel, where $T_{P2P}^{i,k}$ and $T_{NA}^{n,k}$ are the computation times taken for executing (8) for prosumer $i$ and (9) for NA $n$ at iteration $k$.

B. Global vs Marginalized Feasibility Criteria

Participation in P2P energy trading in a particular market interval is a decision up to the individual prosumers, and there is no binding agreement among the prosumers that they must participate in all the market intervals. Hence, the nodal injection amounts are uncertain. Constraint (7) globally satisfies the grid feasibility of the overall P2P prosumer injections, and it is jointly respected by the NAs via coordination. Therewith, Algorithm 2 allows NAs to interact with each other and utilize the network based on the price signals $\lambda$ such that the nodes with highly competitive prosumer participation naturally makes more injections to the grid. Such an approach is less conservative for application in P2P energy trading in comparison to providing a marginalized injection domains as in [22], which restricts the power injections at prosumer nodes independent of the overall injections from the other nodes. This is illustrated below using five scenarios each considering double the number of prosumers while maintaining only a randomly selected 60 active prosumers for P2P energy trading. Further, the same set of prosumer buses were used where NAs are established, and hence, the same constraint parameters $M$ and $r$ are used. The resulting nodal power injections (absolute values) are plotted in Fig. 8, and nodal power injections at prosumer buses are available for all scenarios depending on prosumer participation. It was noted that the sum of absolute values of power injections (that indicates the prosumer hosting capacity) for each scenario is varying with mean and standard deviation of 10.44 MW and 0.30 MW respectively. Therefore, it can be concluded that the proposed market framework satisfies the grid feasibility of nodal power injections, always allows similar hosting capacity regardless of the prosumer location, and allows the P2P market clearing mechanism to decide each individual injections.

V. CONCLUSIONS

In this work, a novel market framework was proposed for P2P energy trading while satisfying the voltage constraints of distribution grids. Here, we constructed polytopic injection regions based on the Kantorovich fixed-point theorem, wherein maintaining the prosumer power injections within those safe/allowed regions guarantees the voltage limits of the distribution grid. We develop convex optimization models for both prosumers and nodal aggregators (established at each
prosumer node of the distribution grid) utilizing the above-mentioned convex injection regions, which support the scalability of the P2P energy market framework. Then, ADMM-based optimization-coordination algorithms were designed for each market participant to execute the market clearing mechanism, and their convergence to global optimality were proven. Case studies were conducted on the modified IEEE 123-bus distribution system locating 60 prosumers in 30 randomly selected buses. The results illustrated that the P2P energy transactions computed without considering their impact on grid constraints lead to infeasible operation. In contrast, the self-validation feature incorporated in the proposed P2P energy market framework satisfies grid constraints while avoiding invalid P2P energy trading contracts and promotes secure and reliable prosumer participation in bilateral energy trading.

REFERENCES


