Partitioning Approach based on Convex Hull and Multiple Choice for Solving Hydro Unit-Commitment Problems

Henderson Gomes e Souza  
Erlon Cristian Finardi*  
Dept. of Electrical Engineering  
Federal University of Santa Catarina  
*and INESC P&D Brasil  
Florianópolis, Brazil

Bruno Henrique Brito  
Federal Institute of Tocantins and  
Federal University of Santa Catarina  
Palmas, Brazil

Fabricio Y. K. Takigawa  
Group of Studies in Energy Systems  
Federal Institute of Santa Catarina  
Florianópolis, Brazil

Abstract — The hydro unit-commitment (HUC) problem aims to determine the status of generating units (GUs) and their generation levels, usually in a day-head planning horizon. The HUC is a complex mixed-integer nonlinear programming (MINLP) problem whose great challenge is accurately representing each hydro production function (HPF) with high computational efficiency. This work proposes to solve the HUC problem using an algorithm based on mixed-integer linear programming (MILP) that updates the gross head iteratively and splits the univariate HPF into concave and convex regions. Piecewise linear approximations are created via the convex hull for concave regions, allowing gathering identical UGs to alleviate the computational burden. On the other hand, the multiple choice model approximates the HPF convex regions. The simulations, carried out in two cascade systems, one with five plants and 18 GUs and the other with four plants and 53 GUs, show that the proposed approach can solve the problem quickly and accurately.

Index Terms — Hydro unit commitment, partitioning of the hydro production function, convex hull, multiple choice.

NOMENCLATURE

Indexes

\( r \)  
hydro plant.

\( i \)  
generating unit (GU).

\( j \)  
group of identical GUs.

\( t \)  
time stage (hour).

\( k \)  
upstream plants of a given plant.

\( n \)  
straight-line used in piecewise linear (PWL) approximation via Convex Hull (CH).

\( m \)  
position of the binary code associated with the number of identical GUs.

\( p \)  
linear segment used in PWL approximation via multiple choice (MC).

Variables

\( q_{rt} \)  
turbined outflow of plant \( r \) in stage \( t \) (m³/s).

\( s_{rt} \)  
spillage of plant \( r \) in stage \( t \) (m³/s).

\( v_{rt} \)  
reservoir volume of plant \( r \) at the beginning of stage \( t \) (hm³).

\( v_{\text{ave}} \)  
average volume of reservoir \( r \) in stage \( t \) (hm³).

\( g_{ijrt} \)  
power generation of GU \( i \), group \( j \), plant \( r \), and stage \( t \) (MW).

\( \eta_{ijrt} \)  
efficiency of GU \( i \), group \( j \), hydro \( r \), and stage \( t \).

\( h_r \)  
gross head of plant \( r \) in stage \( t \) (m).

\( w_{ijrt} \)  
turbined outflow of GU \( i \), group \( j \), plant \( r \), and stage \( t \) (m³/s).

\( u_{ijr} \)  
binary variable that indicates whether GU \( i \) of group \( j \) and plant \( r \) is operating (\( u_{ijr} = 1 \)) or not (\( u_{ijr} = 0 \)) in stage \( t \).

\( c_{ijr} \)  
binary variable that indicates if GU \( i \) of group \( j \) and plant \( r \) is turned on (\( c_{ijr} = 1 \)) in stage \( t \).

\( o_{ijr} \)  
variable that indicates if GU \( i \) of group \( j \) and plant \( r \) is shut down (\( o_{ijr} = 1 \)) in stage \( t \).

\( g_{cijr} \)  
generation in the concave region of GU \( i \), group \( j \), plant \( r \) and stage \( t \) (MW).

\( g_{vijr} \)  
generation in the convex region of GU \( i \), group \( j \), plant \( r \) and stage \( t \) (MW).

\( w_{cijr} \)  
turbined outflow in the concave region of GU \( i \), group \( j \), plant \( r \) and stage \( t \) (m³/s).

\( w_{vijr} \)  
turbined outflow in the convex region of GU \( i \), group \( j \), plant \( r \) and stage \( t \) (m³/s).

\( u_{ci} \)  
binary variable that indicates whether GU \( i \), group \( j \) and plant \( r \) is operating (\( u_{ci} = 1 \)), or not (\( u_{ci} = 0 \)), in its concave region in stage \( t \).

\( u_{vi} \)  
binary variable that indicates whether GU \( i \), group \( j \) and plant \( r \) is operating (\( u_{vi} = 1 \)), or not (\( u_{vi} = 0 \)), in its convex region in stage \( t \).

\( y_{mi} \)  
binary variable \( m \) of the binary code representing the number of GUs of plant \( r \)’s group \( j \) operating in stage \( t \).

\( w_{sijr} \)  
turbined outflow of GUs of group \( j \), plant \( r \), and stage \( t \) (m³/s).

\( g_{sijr} \)  
generation of GUs of group \( j \), plant \( r \), and stage \( t \) (MW).

\( r_{wijk} \)  
turbined outflow auxiliary variable of group \( j \), plant \( r \) in stage \( t \) (m³/s).
Due to the increasing insertion of intermittent renewable energy in power systems worldwide, hydro plants play a vital role in addressing the uncertainties. Consequently, accurate modeling and efficient solution techniques of the hydro unit-commitment (HUC) problem, which aims to efficiently determine the status of generating units (GUs) and their generation levels, usually in a day-head planning horizon, are critical in ensuring the best use of the hydro dispatch flexibility to handle the uncertainties. The central modeling issue in HUC is the precise representation of the nonlinear nonconvex hydro production function (HPF), which depends on the head and turbine outflow [1]. However, such representation confronts significant modeling and computational challenges. As the HUC is a large-scale mixed-integer nonlinear programming (MINLP) problem, several strategies based on mathematical programming and heuristics have already been tested to find optimal solutions [2]. The aim is to improve the tradeoff between solution quality and computational performance.

In recent years, approaches based on mixed-integer linear programming (MILP) have gained prominence [1],[2] due to efficient commercial solvers, such as CPLEX and GUROBI. There are two typical approaches to model HPF in HUC problems when handled via MILP-based strategies. In the first approach, the HPF is represented by mixed integer piecewise linear (PWL) constraints that can precisely account for the nonlinearities and nonconvexities [3]-[9]. This approach is appealing because commercial solvers can efficiently solve MILP problems consistently and reliably. However, it always faces a tradeoff: whereas the accuracy can be enhanced with a larger number of HPF breakpoints, the computational burden also increases due to the increasing number of required binary
variables. In the second approach, the HPF is replaced by its convex hull (CH), where the PWL approximation does not include binary variables [10]-[14]. However, the CH may suffer from accuracy issues when approximating nonconcave regions. Furthermore, the accuracy of the CH cannot be improved by increasing the number of constraints. As this approach relaxes the constraint set of the HUC, there is no guarantee that the solution is still feasible w.r.t. the original HPF.

A further distinction between MILP-based approaches to solving HUC is how the net head effect is incorporated into the HPF. In some works, this issue is directly incorporated into the MILP problem via bivariate HPF [6]-[9], [12], [14]. In contrast, a set of univariate HPFs is linearized in other works, and binary variables are introduced to select the PWL approximation of a univariate HPF associated with the most suitable net head [5], [15]. Finally, some approaches update the head effects dynamically, solving a problem that considers the PWL approximation of a univariate HPF associated with the head at that iteration [13], [16].

In this scenario, an important challenge for solving the HUC via MILP is, even for systems with many UGs, representing precisely the nonconvexities of the HPF without increasing time since the HUC possesses limited computational constraints. In an attempt to fill this gap, inspired by disjunctive programming techniques [17]-[19], this paper proposes partitioning the domain of the HPF in the regions in which it is convex and those in which it is concave. By considering a univariate HPF, the domain partitioning is achieved using the second derivative to determine the number of identical GUs instead of finding different PWL of each partitioned region.

Thus, the domain partitioning is achieved using the second derivative to determine the number of identical GUs instead of finding different PWL of each partitioned region. However, the partitioning approach is not suitable in convex regions of the HPF. In this scenario, the HPF is replaced by its convex hull (CH), where the PWL approximation does not include binary variables [10]-[14]. However, the CH may suffer from accuracy issues when approximating nonconcave regions. Furthermore, the accuracy of the CH cannot be improved by increasing the number of constraints. As this approach relaxes the constraint set of the HUC, there is no guarantee that the solution is still feasible w.r.t. the original HPF.

To conclude, this paper is organized as follows: Section II presents the mixed-integer nonlinear programming formulation of the HUC problem; in Section III, the proposed algorithm that considers a partitioning-based approach is detailed; Section IV shows the results related to two cascade configurations extracted from the Brazilian power system. Then, the final remarks are given in Section V.

II. THE HUC PROBLEM

The solution strategy proposed in this paper considers a HUC problem for a centrally dispatched, cost-minimization-based market. In this market framework, each plant receives hourly generation targets for the next day from the Independent System Operator (ISO) — hydro plants must meet these targets while efficiently using the water available to them [1]. Thus, the MINLP model of the HUC problem considered in this work is given in (1)-(16). Since clustering is used to cope with identical GUs, an index $j$ is included to identify groups of identical units in each hydro plant $r$.

$$
\min \Theta = \sum_{r=1}^{R} \sum_{t=1}^{T} (q_{rt} + s_{rt}) + \sum_{r=1}^{R} \sum_{j=1}^{J_r} \sum_{i=1}^{I_r} \sum_{u_{ij}} [S_j \cdot (e_{gu} + o_{gu})]
$$

s.t.

$$
v_{r,t} - v_{r,t} + H \cdot [q_{rt} + s_{rt} - \sum_{k=1}^{K} (q_{k,t-\tau} + s_{k,t-\tau})] = 0,
$$

$$
\forall r, \forall t
$$

$$
V_{min} \leq V_{r,t} \leq V_{max}, \forall r, \forall t
$$

$$
0 \leq s_{rt} \leq S_{max}, \forall r, \forall t
$$

$$
g_{fr} - 9.81 \cdot 10^{-3} \cdot \eta_{fr} \cdot (h_{t} - D_{fr} \cdot w_{fr}^3) \cdot w_{fr} = 0,
$$

$$
\forall i, \forall j, \forall r, \forall t
$$

$$
\eta_{fr} - [A_{fr} + A_{fr} \cdot \omega_{fr} + A_{fr} \cdot \omega_{fr}^2] + A_{fr} \cdot \omega_{fr}^3 + A_{fr} \cdot \omega_{fr}^4 + A_{fr} \cdot \omega_{fr}^5 = 1, \forall i, \forall j, \forall r, \forall t
$$

$$
h_{fr} - (B_{fr} + B_{fr} \cdot \omega_{fr} + B_{fr} \cdot \omega_{fr}^2 + B_{fr} \cdot \omega_{fr}^3 + B_{fr} \cdot \omega_{fr}^4 + B_{fr} \cdot \omega_{fr}^5) + (C_{fr} + C_{fr} \cdot (q_{fr} + s_{fr}) + C_{fr} \cdot (q_{fr} + s_{fr})^2 + C_{fr} \cdot (q_{fr} + s_{fr})^3 + C_{fr} \cdot (q_{fr} + s_{fr})^4 = 0,
$$

$$
\forall i, \forall j, \forall r, \forall t
$$

$$
v_{fr} = \frac{v_{fr} + v_{fr-1}}{2}, \forall r, \forall t
$$

$$
W_{fr} \cdot u_{fr} \leq W_{fr} \cdot u_{fr}, \forall i, \forall j, \forall r, \forall t
$$

$$
C_{fr} \cdot u_{fr} \leq C_{fr} \cdot u_{fr}, \forall i, \forall j, \forall r, \forall t
$$

$$
\sum_{i=1}^{I_r} \sum_{j=1}^{J_r} g_{u_{ij}} - L_{rt} = 0, \forall r, \forall t
$$
\[ \sum_{i=1}^{l_p} \sum_{j=1}^{l_j} w_{ijr} - q_{ir} = 0, \forall r, \forall t \]  
(12)

\[ e_{ijr} - u_{ijr} + u_{ijr,t-1} \geq 0, \forall i, \forall j, \forall r, \forall t \]  
(13)

\[ o_{ijr} - u_{ijr,t-1} + u_{ijr} \geq 0, \forall i, \forall j, \forall r, \forall t \]  
(14)

\[ e_{ijr} \geq 0, o_{ijr} \geq 0, \forall i, \forall j, \forall r, \forall t \]  
(15)

\[ u_{ijr} \in \{0,1\}, \forall i, \forall j, \forall r, \forall t. \]  
(16)

The objective function (1) aims to minimize the sum of the total outflow of plants and the switching of GUs, which are penalized with \( SS_r \). The adjustment of objective weights, \( SS_r \), is based on the order of magnitude between the two terms: the plant turbinated outflow and the remaining parts responsible for including the start-ups and shut-downs in the GUs.

Concerning the feasible set, constraints (2) ensure the water balance in the reservoirs. Constraints (3) and (4) represent reservoir volume and spillage limits. The HPF of each GU is represented by (5), and it involves the generation, efficiency (6), gross head (7), and the turbinated outflow of the GU. The term \( D_{jr} \cdot w_{ijr}^2 \) in (5) and (6) accounts for the head loss in the penstock. Therefore, the function \( (h_{ir} - D_{jr} \cdot w_{ijr}^2) \) provides the net head of GU \( i \), group \( j \), plant \( r \), and stage \( t \). The average volume in stage \( t \) is defined in (8). Constraints (9) and (10) bound the turbinated outflow and power generation of each GU, respectively. The constraints that enforce meeting the generation targets of each plant in each time stage are given by (11). Constraint (12) represents the water balance in the penstocks. Constraints (13)-(15) indicate when a GU is started up or shut down. Finally, (16) are the integrality constraints.

### III. Proposed Strategy

The first step to describe the proposed strategy consists in modifying (1)-(16), assuming that the gross head of each plant (7) is known in each time stage. Equation (5)-(7) can be replaced by the following univariate HPF once the gross head is known.

\[ g_{ijr} = 9.81 \cdot 10^{-3} \cdot [A_{jr} + A_{jr} \cdot w_{ijr} + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \]

\[ + A_{jr} \cdot w_{ijr}^2 + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2) - D_{jr} \cdot w_{ijr}^2 \cdot w_{ijr} = 0, \forall i, \forall j, \forall r, \forall t \]  
(17)

In the equation, \( GH_{ir} \) is the estimate of the gross head of plant \( r \) in stage \( t \), which is obtained as follows:

\[ GH_{ir} = (B_{r0} + B_{r1} \cdot v_{ir} + B_{r2} \cdot v_{ir}^2 + B_{r3} \cdot v_{ir}^3 + B_{r4} \cdot v_{ir}^4) \]

\[ - [C_{r0} + C_{r1} \cdot (q_{ir} + s_{ir}) + C_{r2} \cdot (q_{ir} + s_{ir})^2 + C_{r3} \cdot (q_{ir} + s_{ir})^3 + C_{r4} \cdot (q_{ir} + s_{ir})^4], \forall r, \forall t \]  
(18)

The next step consists in partitioning the univariate HPFs (17) into concave and convex regions. Then, to find the inflection points of the HPF, it is necessary to find the roots of the second derivative of (17). Fig. 1 illustrates the inflection point that separates a univariate HPF into a convex and a concave region.

Next, after finding the inflection points, (9) and (17) are rewritten below. Typically, a univariate HPF is composed of a convex region and a concave region, as illustrated in (13) and Fig. 1. Therefore, for simplicity, we consider that each HPF has a single concave region and a single convex region, i.e., there is only one inflection point for each operating zone.

\[ g_c_{ijr} = 9.81 \cdot 10^{-3} \cdot [A_{jr} + A_{jr} \cdot w_{ijr} + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2) \]

\[ + A_{jr} \cdot w_{ijr}^2 + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2) - D_{jr} \cdot w_{ijr}^2 \cdot w_{ijr} = 0, \forall i, \forall j, \forall r, \forall t \]  
(19)

\[ W_{jr} \cdot wc_{ijr} \leq wc_{ijr} \leq W_{jr}^{max} \cdot wc_{ijr}, \forall i, \forall j, \forall r, \forall t \]  
(20)

\[ g_v_{ijr} = 9.81 \cdot 10^{-3} \cdot [A_{jr} + A_{jr} \cdot w_{ijr} + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2) \]

\[ + A_{jr} \cdot w_{ijr}^2 + A_{jr} \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2)] \cdot (GH_{ir} - D_{jr} \cdot w_{ijr}^2) - D_{jr} \cdot w_{ijr}^2 \cdot w_{ijr} = 0, \forall i, \forall j, \forall r, \forall t \]  
(21)

\[ W_{jr}^{max} \cdot uv_{ijr} \leq uv_{ijr} \leq W_{jr}^{max} \cdot uv_{ijr}, \forall i, \forall j, \forall r, \forall t \]  
(22)

\[ g_{ijr} + g_{ijr} = g_{ijr}, \forall i, \forall j, \forall r, \forall t \]  
(23)

\[ w_{ijr} + w_{ijr} = w_{ijr}, \forall i, \forall j, \forall r, \forall t \]  
(24)

\[ u_{ijr} + u_{ijr} = u_{ijr}, \forall i, \forall j, \forall r, \forall t \]  
(25)

\[ u_{ijr} + u_{ijr} \in \{0,1\}, \forall i, \forall j, \forall r, \forall t. \]  
(26)

Constraints (19)-(20) represent the HPF model in the concave region, while (21)-(22) is the HPF model in the convex region. In turn, auxiliary constraints (23)-(26) ensure that a GU does not operate simultaneously in two regions.

The next step consists of linearizing the problem given by (1)-(4), (10)-(16), (19)-(26) to obtain a MILP model, whose final formulation can be found in section III-C. The
linearization steps of the concave and convex regions are
detailed in sections III-A and III-B, respectively. Once the
MILP shown in section III-C has been solved, the solutions \( V_{rt} \),
\( W_{ijrt} \), and \( S_{rt} \) are used to estimate new values of \( G_H \), through
(8), (12), and (18). For the first iteration, values close to the
current operating conditions of the plant can be estimated.
Subsequently, the new values of \( G_H \) are used in (19) and (21),
and a new linearization process is performed. Then, the
algorithm stops when the difference between the objective
function values in the last two iterations is not greater than a
given tolerance. Other possibilities of stopping criteria are
discussed in [13], highlighting that the algorithm normally
stops between three and five iterations. The proposed strategy
for solving the problem (1)-(16) is summarized in the
flowchart in Fig. 2.

A. Linearizing the concave region

One of the advantages of partitioning the HPF is using a
PWL approximation via CH in the concave region. This
approach is computationally appealing for avoiding binary
variables to identify the straight line approximating the original
nonlinear function. Consequently, this approximation can be
enhanced by increasing the number of turbine outflow and
power generation points \([W_{ijrt}, G_C^{ijrt}]\), where \( v \in \text{CH} \)
are points chosen to approximate (19). Thus, PWL approximation
via CH is given in (27).

\[
g_{ijrt}^c \leq E_{0,ijrt}^n \cdot G_{ijrt}^c + E_{ijrt}^n \cdot W_{ijrt}^c,
\quad n = 1, \ldots, NH_{ijrt}, \forall i, \forall j, \forall r, \forall t
\]

Figure 2. Flowchart of the proposed strategy.

Since identical GUs in group \( j \) of plant \( r \) operating in the HPF’s
concave region in stage \( t \) Thus, applying the clustering
techniques used in [20], all constraints that form the CH of
group \( j \) can be replaced by:

\[
\sum_{m=1}^{M_j} (2^{n-1} \cdot G_{ijrt}^m) \leq E_{0,ijrt}^n \cdot \sum_{m=1}^{M_j} (2^{n-1} \cdot Y_{ijrt}^m) +
E_{ijrt}^n \cdot \sum_{m=1}^{M_j} (2^{n-1} \cdot W_{ijrt}^m), \forall n, \forall j, \forall r, \forall t
\]

(28)

\[
W_{ijrt}^c \cdot Y_{ijrt}^m \leq W_{ijrt}^c \cdot Y_{ijrt}^m, \forall m, \forall j, \forall r, \forall t
\]

(29)

\[
E_{ijrt}^n \cdot (W_{ijrt}^c - Y_{ijrt}^m) \leq W_{ijrt}^c \cdot Y_{ijrt}^m, \forall m, \forall j, \forall r, \forall t
\]

(30)

\[
G_{ijrt}^c \cdot Y_{ijrt}^m \leq G_{ijrt}^c \cdot Y_{ijrt}^m, \forall m, \forall j, \forall r, \forall t
\]

(31)

\[
G_{ijrt}^c \cdot (W_{ijrt}^c - Y_{ijrt}^m) \leq G_{ijrt}^c \cdot Y_{ijrt}^m, \forall m, \forall j, \forall r, \forall t
\]

(32)

For instance, whereas 70 constraints, ten continuous
variables, and five binary ones are necessary to represent the
individual HPFs of five identical GUs with ten straight lines via
the CH, the symmetric representation requires only 35
constraints (-50%), eight continuous variables (-20%) and four
binary variables (-20%). This difference raises as the number of
identical GUs grows.

B. Linearizing the convex region

It is necessary to use a model with binary variables for the
PWL approximation in the convex region. Additionally, since the
dispatch in the convex region can be different even for
identical GUs in an optimal solution, each HPF needs to be
approximated individually. Thus, this paper considers
approximating the convex region of each GU via the MC model
[24], [25], replacing (21) and (22) with the following
constraints:

\[
w_{ijrt} = \sum_{p=1}^{P} w_{ijrt}^p, \forall i, \forall j, \forall r, \forall t
\]

(35)

\[
g_{ijrt}^c = \sum_{p=1}^{P} (E_{0,ijrt}^p \cdot w_{ijrt}^p + F_{ijrt}^p \cdot z_{ijrt}^p), \forall i, \forall j, \forall r, \forall t
\]

(36)

\[
w_{ijrt}^\text{min}, p \cdot z_{ijrt}^p \leq w_{ijrt}^p \leq w_{ijrt}^\text{max}, p, \forall p, \forall i, \forall j, \forall r, \forall t
\]

(37)

\[
\sum_{p=1}^{P} z_{ijrt}^p = v_{ijrt}, \forall i, \forall j, \forall r, \forall t
\]

(38)

C. MILP formulation of the HUC problem

This section presents the MILP used in each algorithm
iteration presented in Fig. 2. In addition, modifications in the
objective function and the feasible set are needed to deal with
start-ups and shutdowns in the proposed formulation since the total number of GUs being started up/shutdown in each group and time stage must be considered. Therefore, the formulation to solve the HUC problem via MILP in each iteration of the algorithm is given by:

\[
\min \Theta = \sum_{r=1}^{R} \sum_{t=1}^{T} (q_{rt} + s_{rt}) + \sum_{j=1}^{J} \sum_{l=1}^{L} \sum_{r=1}^{R} \sum_{t=1}^{T} [S_{rt} \cdot (s_{rt} + q_{rt})] \quad (39)
\]

s.t.: \(2)-(4), (28)-(38)\)

\[
\sum_{j=1}^{J} \sum_{m=1}^{M} (2^{m-1} \cdot r_{g_{mt}}) - L_{rt} = 0, \forall r, \forall t \quad (41)
\]

\[
\sum_{j=1}^{J} \sum_{m=1}^{M} (2^{m-1} \cdot w_{g_{mt}}) - q_{rt} = 0, \forall r, \forall t \quad (42)
\]

\[
\sum_{i=1}^{I} r_{g_{mt}} + \sum_{m=1}^{M} (2^{m-1} \cdot y_{g_{mt}}) = d_{g_{mt}}, \forall j, \forall r, \forall t \quad (43)
\]

\[
se_{rt} - d_{rt} + d_{r,t-1} \geq 0, \forall j, \forall r, \forall t \quad (44)
\]

\[
so_{rt} - d_{r,t-1} + d_{r,t} \geq 0, \forall j, \forall r, \forall t \quad (45)
\]

\[
se_{rt} \geq 0, \forall j, \forall r, \forall t \quad (46)
\]

IV. COMPUTATIONAL RESULTS

In this section, the performance of the proposed algorithm is analyzed in two cascade systems: the first is composed of five plants and 18 GUs, and the second is composed of four plants and 53 GUs. The planning horizon extends to a single day with hourly discretization. To accurately represent the nonlinearities of the HPF, the simulations consider 50 equidistant points in the concave region and five in the convex region. The finer discretization in the concave region is due to three reasons: (i) it is the major region in the HPF; (ii) since it uses CH, it requires a smaller number of mixed-integer constraints when compared to the MC model; and, (iii) it is the region where the maximum efficiency is located. Finally, to assess penalization for start-ups and shutdowns, the simulations consider \(SS_{ij}\) assuming 0%, 25%, 50%, and 100% of \(W_{ij}^{max}\).

To validate the proposed algorithm, which is called PROP, we compare with two other MILP approaches that consider two ways to approximate the HPF without applying clustering in each iteration. In the first one, called CH, a concave approximation is applied to each univariate HPF, as in [13]. In this way, the problem to be solved in each iteration is given by (1)-(4), (9)-(16) and (48).

\[
g_{g_{rt}} \leq E_{0g_{rt}} \cdot u_{g_{rt}} + E_{1g_{rt}} \cdot w_{g_{rt}}, \forall n \in NH_{g_{rt}}, \forall i, \forall j, \forall r, \forall t \quad (48)
\]

Fig. 3 shows the straight lines generated by CH and PROP by using nine points in the concave region and four in the convex region. Note that, while PROP captures the nonconvexity, the CH creates a CH over the HPF, generating a superior linearization error mainly in the convex region.

In the second strategy, called MC, a nonconvex approximation via MC is applied to the univariate HPF. Therefore, the problem solved in each iteration in MC is given by (1)-(4), (9)-(16), and (49)-(52).

\[
w_{g_{rt}} = \sum_{p=1}^{P} w_{g_{rt}}^{p}, \forall t, \forall j, \forall r, \forall t \quad (49)
\]

\[
g_{g_{rt}} = \sum_{p=1}^{P} (F_{0g_{rt}}^{p} \cdot w_{g_{rt}}^{p} + F_{1g_{rt}}^{p} \cdot z_{g_{rt}}^{p}), \forall i, \forall j, \forall r, \forall t \quad (50)
\]

\[
W_{ij}^{min} \cdot z_{ij}^{p} \leq w_{g_{rt}}^{p} \leq W_{ij}^{max} \cdot z_{ij}^{p}, \forall p, \forall i, \forall j, \forall r, \forall t \quad (51)
\]

\[
\sum_{p=1}^{P} w_{g_{rt}}^{p} = u_{g_{rt}}, \forall i, \forall j, \forall r, \forall t \quad . \quad (52)
\]

The coefficients \(E_{0g_{rt}}^{p}\) and \(E_{1g_{rt}}^{p}\) in CH and MC are obtained considering the same discretization methodology of PROP. The results for the system with 18 GUs will be presented in more detail. For this system, the PROP is also compared with the formulation that considers the bivariate approximation of HPF via the Logarithmic Aggregated Convex Combination model, called LACC [9][23]. However, in the number of triangles to represent the bivariate HPF negatively affects the computational time in systems with many GUs. In this context, the idea is to compare the results obtained with different discretizations of the bivariate HPF via LACC and the PROP results.

The MILPs were solved with GUROBI 9.5.0 via PYTHON 3.7. The computer has an Intel Core i7-1165g7 4.60 GHz processor and 8 GB RAM. As for the stopping criteria, a tolerance of 0.01% between the objective function values in the last two iterations is used. In the MILP solved at each iteration, a relative gap tolerance of 0.5% is set at the first iteration, then tightened to 0.05% at the second iteration and 0% for the remaining iterations.

A. Results for the cascade with five plants and 18 GUs

The hydraulic configuration of this system can be seen in Fig. 4.
Table 1 presents, for each hydro plant, the power capacity, number of GUs, per-UG limits on turbined outflow and power generation, as well as the limits on reservoir volume. In this system, all units in the same plant are identical. Furthermore, the water traveling time between two coupled reservoirs is one hour.

Table 1 – Main operational characteristics of the plants of the system with 18 GUs

<table>
<thead>
<tr>
<th>Plant</th>
<th>Capacity (MW)</th>
<th>I₀</th>
<th>H_{min} (m³/s)</th>
<th>G_{min} (MW)</th>
<th>P_{max} (hm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>600</td>
<td>3</td>
<td>50/177</td>
<td>70/230</td>
<td>2,700/4,900</td>
</tr>
<tr>
<td>#2</td>
<td>870</td>
<td>3</td>
<td>60/186</td>
<td>80/290</td>
<td>1,320/1,477</td>
</tr>
<tr>
<td>#3</td>
<td>1,140</td>
<td>3</td>
<td>80/430</td>
<td>120/380</td>
<td>2,280/3,340</td>
</tr>
<tr>
<td>#4</td>
<td>1,450</td>
<td>5</td>
<td>70/318</td>
<td>70/290</td>
<td>4,300/5,100</td>
</tr>
<tr>
<td>#5</td>
<td>860</td>
<td>4</td>
<td>50/470</td>
<td>40/215</td>
<td>1,430/1,500</td>
</tr>
</tbody>
</table>

On the other hand, Table 2 shows the coefficients of the net-head function for each plant. All GUs have one concave and one convex region.

The coefficients $A_0$, of the efficiency functions are omitted due to confidentiality requirements. The hourly generation targets are presented in Fig. 5.

![Figure 5. Generation targets for the five plants.](image)

Table 2 - Initial volumes and incremental inflows for the five plants

<table>
<thead>
<tr>
<th>Plant</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
</tr>
</thead>
<tbody>
<tr>
<td>V₀</td>
<td>3,731</td>
<td>1,360</td>
<td>2,766</td>
<td>4,856</td>
<td>1,458</td>
</tr>
<tr>
<td>Yᵣ</td>
<td>100</td>
<td>130</td>
<td>550</td>
<td>400</td>
<td>100</td>
</tr>
</tbody>
</table>

Next, the results of PROP are compared with MC and CH approaches. Subsequently, the results of PROP are compared with the LACC bivariate approach.

I) Comparing PROP with MC and CH

Table 3 presents the squared root of the average relative quadratic error in the concave (MEC) and convex (MEV) regions, and the maximum error over the entire domain (MXE), for each plant considering the minimum net head ($H_{min}$) and the maximum ($H_{max}$). The errors were obtained based on comparing the piecewise linear and the nonlinear HPF power generations obtained with 0.01-m³/s steps in the turbined outflow feasible domain. The linearization errors are significantly smaller in the PROP and MC strategies, especially in the convex region.

Table 3 – Error in the PWL approximation of the HPF in the PROP, CH, and MC strategies

<table>
<thead>
<tr>
<th>Plant</th>
<th>Strategy</th>
<th>$H_{min}$ MEV</th>
<th>$H_{min}$ MEC</th>
<th>$H_{max}$ MEV</th>
<th>$H_{max}$ MEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>PROP/MC</td>
<td>0.037</td>
<td>0.003</td>
<td>0.085</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>CH</td>
<td>0.635</td>
<td>0.059</td>
<td>0.828</td>
<td>0.036</td>
</tr>
<tr>
<td>#2</td>
<td>PROP/MC</td>
<td>0.055</td>
<td>0.005</td>
<td>0.126</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>CH</td>
<td>1.095</td>
<td>0.015</td>
<td>1.407</td>
<td>0.275</td>
</tr>
<tr>
<td>#3</td>
<td>PROP/MC</td>
<td>0.173</td>
<td>0.010</td>
<td>0.412</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>CH</td>
<td>2.567</td>
<td>0.067</td>
<td>3.440</td>
<td>0.058</td>
</tr>
<tr>
<td>#4</td>
<td>PROP/MC</td>
<td>0.175</td>
<td>0.004</td>
<td>0.416</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>CH</td>
<td>2.626</td>
<td>0.011</td>
<td>3.513</td>
<td>0.044</td>
</tr>
<tr>
<td>#5</td>
<td>PROP/MC</td>
<td>0.246</td>
<td>0.016</td>
<td>0.601</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>CH</td>
<td>5.079</td>
<td>0.017</td>
<td>6.336</td>
<td>0.068</td>
</tr>
</tbody>
</table>

The main results of strategies PROP, CH, and MC, for different values of $S_x$, are presented in Table 4. For each simulation, the value of the objective function (OBJ), the number of start-ups and shutdowns of the GUs in the cascade (NSS), the computing time (ST) in seconds, and the number of iterations (IT) are presented. The differences between CH and PROP are presented in the rows identified by DIF.CH, while the differences between MC and PROP are in DIF.MC.

Table 4 – Main results in system with 18 GUs

<table>
<thead>
<tr>
<th>Plant</th>
<th>OBJ $S_x$</th>
<th>NSS 0%</th>
<th>NSS 25%</th>
<th>NSS 50%</th>
<th>NSS 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CH</td>
<td>91,108.1</td>
<td>92,358.8</td>
<td>92,753.8</td>
<td>92,889.9</td>
<td></td>
</tr>
<tr>
<td>MC</td>
<td>91,154.0</td>
<td>92,639.3</td>
<td>93,041.8</td>
<td>93,319.8</td>
<td></td>
</tr>
<tr>
<td>PROP</td>
<td>91,149.5</td>
<td>92,636.3</td>
<td>93,039.8</td>
<td>93,315.4</td>
<td></td>
</tr>
<tr>
<td>DIF.CH</td>
<td>0.045%</td>
<td>0.300%</td>
<td>0.308%</td>
<td>0.458%</td>
<td></td>
</tr>
<tr>
<td>DIF.MC</td>
<td>-0.005%</td>
<td>-0.003%</td>
<td>-0.002%</td>
<td>-0.005%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Plant</th>
<th>ST 0%</th>
<th>ST 25%</th>
<th>ST 50%</th>
<th>ST 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CH</td>
<td>10.51</td>
<td>13.67</td>
<td>26.61</td>
<td>20.61</td>
</tr>
<tr>
<td>MC</td>
<td>167.73</td>
<td>221.23</td>
<td>204.92</td>
<td>219.65</td>
</tr>
<tr>
<td>PROP</td>
<td>19.67</td>
<td>24.04</td>
<td>21.21</td>
<td>19.33</td>
</tr>
<tr>
<td>DIF.CH</td>
<td>87%</td>
<td>76%</td>
<td>-20%</td>
<td>-6%</td>
</tr>
<tr>
<td>DIF.MC</td>
<td>-88%</td>
<td>-89%</td>
<td>-90%</td>
<td>-91%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Plant</th>
<th>IT 0%</th>
<th>IT 25%</th>
<th>IT 50%</th>
<th>IT 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CH</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>MC</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>PROP</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>DIF.CH</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>DIF.MC</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The linearization errors are significantly smaller in the PROP and MC strategies, especially in the convex region.
Observing the results from Table 4, initially, note that the values OBJ in strategy PROP are similar to MC and larger than CH in all simulations. The smaller values of CH occur due to the underestimate of the turbined outflows necessary to meet the generation targets caused by the concave approximation of the HPFs. Since the linearization errors in CH are larger in the convex region, usually associated with low levels of turbined outflow, the differences in OBJ tend to increase with \( W_{\text{max}} \), because more GUs need to operate at low generation levels. Table 6 gives, via NSCV, the number of stages in which each plant has at least one GU operating in the convex region for strategy PROP. Additionally, DIF_WCC (DIF_WCV) gives the average difference between the turbined outflow obtained with CH and PROP in the stages where all GUs operate in the concave (convex) region. Note that DIF_WCV is significantly larger than DIF_WCC, which indicates that CH loses precision when GUs need to be dispatched in the convex region.

Table 6 – Differences in turbined outflow in the concave and convex regions between CH and PROP

<table>
<thead>
<tr>
<th>( W_{\text{max}} )</th>
<th>0%</th>
<th>25%</th>
<th>50%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSCV (12+3+5+15+65)</td>
<td>0+0+0+0+0</td>
<td>3+9+2+4+10</td>
<td>3+12+2+6+10</td>
<td>3+22+2+11+10</td>
</tr>
<tr>
<td>DIF_WCC (m/s)</td>
<td>0.072</td>
<td>0.229</td>
<td>0.105</td>
<td>0.046</td>
</tr>
<tr>
<td>DIF_WCV (m/s)</td>
<td>0.607</td>
<td>2.097</td>
<td>1.956</td>
<td>1.994</td>
</tr>
</tbody>
</table>

Table 8 shows the relative differences in turbined outflow of each plant resulting from PROP and CH for the simulation with the highest penalty for start-ups and shutdowns (\( \alpha = 100\% \) of \( W_{\text{max}} \)). All GUs are dispatched in the convex region in the cells highlighted in yellow. On the other hand, at least one GU operates in the convex region in those highlighted in green. Note that as the number of GUs operating in the convex region increases, the difference in turbined outflow and, consequently, the error of CH is increased.

Table 7 – Differences in turbined outflow between PROP and CH for a simulation with \( \alpha \) assuming 100% of \( W_{\text{max}} \)

<table>
<thead>
<tr>
<th>Stage</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000%</td>
<td>0.616%</td>
<td>0.000%</td>
<td>2.154%</td>
<td>3.516%</td>
</tr>
<tr>
<td>2</td>
<td>0.000%</td>
<td>1.909%</td>
<td>0.000%</td>
<td>3.518%</td>
<td>5.277%</td>
</tr>
<tr>
<td>3</td>
<td>0.987%</td>
<td>2.854%</td>
<td>0.002%</td>
<td>4.400%</td>
<td>5.279%</td>
</tr>
<tr>
<td>4</td>
<td>0.747%</td>
<td>3.035%</td>
<td>2.637%</td>
<td>1.725%</td>
<td>1.999%</td>
</tr>
<tr>
<td>5</td>
<td>1.344%</td>
<td>4.312%</td>
<td>2.501%</td>
<td>1.483%</td>
<td>2.611%</td>
</tr>
<tr>
<td>6</td>
<td>1.391%</td>
<td>2.112%</td>
<td>0.036%</td>
<td>0.857%</td>
<td>0.977%</td>
</tr>
<tr>
<td>7</td>
<td>0.000%</td>
<td>1.517%</td>
<td>0.003%</td>
<td>0.488%</td>
<td>0.003%</td>
</tr>
<tr>
<td>8</td>
<td>0.000%</td>
<td>1.499%</td>
<td>0.004%</td>
<td>0.044%</td>
<td>0.003%</td>
</tr>
<tr>
<td>9</td>
<td>0.001%</td>
<td>0.577%</td>
<td>0.006%</td>
<td>0.002%</td>
<td>0.002%</td>
</tr>
<tr>
<td>10</td>
<td>0.017%</td>
<td>0.527%</td>
<td>0.004%</td>
<td>0.003%</td>
<td>0.002%</td>
</tr>
<tr>
<td>11</td>
<td>0.107%</td>
<td>0.309%</td>
<td>0.005%</td>
<td>0.005%</td>
<td>0.003%</td>
</tr>
<tr>
<td>12</td>
<td>0.000%</td>
<td>1.094%</td>
<td>0.007%</td>
<td>0.012%</td>
<td>0.014%</td>
</tr>
<tr>
<td>13</td>
<td>0.000%</td>
<td>2.700%</td>
<td>0.006%</td>
<td>0.011%</td>
<td>2.608%</td>
</tr>
<tr>
<td>14</td>
<td>0.000%</td>
<td>3.098%</td>
<td>0.000%</td>
<td>0.056%</td>
<td>0.313%</td>
</tr>
<tr>
<td>15</td>
<td>0.000%</td>
<td>3.905%</td>
<td>0.014%</td>
<td>1.864%</td>
<td>0.021%</td>
</tr>
<tr>
<td>16</td>
<td>0.000%</td>
<td>1.330%</td>
<td>0.038%</td>
<td>2.066%</td>
<td>0.002%</td>
</tr>
<tr>
<td>17</td>
<td>0.021%</td>
<td>1.322%</td>
<td>0.011%</td>
<td>0.433%</td>
<td>0.001%</td>
</tr>
<tr>
<td>18</td>
<td>0.022%</td>
<td>0.924%</td>
<td>0.028%</td>
<td>0.016%</td>
<td>0.016%</td>
</tr>
<tr>
<td>19</td>
<td>0.075%</td>
<td>0.755%</td>
<td>0.015%</td>
<td>0.001%</td>
<td>0.005%</td>
</tr>
<tr>
<td>20</td>
<td>0.010%</td>
<td>0.006%</td>
<td>0.013%</td>
<td>0.002%</td>
<td>0.004%</td>
</tr>
<tr>
<td>21</td>
<td>0.000%</td>
<td>0.542%</td>
<td>0.010%</td>
<td>0.001%</td>
<td>0.004%</td>
</tr>
<tr>
<td>22</td>
<td>0.026%</td>
<td>0.917%</td>
<td>0.009%</td>
<td>0.005%</td>
<td>0.890%</td>
</tr>
<tr>
<td>23</td>
<td>0.282%</td>
<td>1.388%</td>
<td>0.017%</td>
<td>0.003%</td>
<td>1.944%</td>
</tr>
<tr>
<td>24</td>
<td>0.002%</td>
<td>3.787%</td>
<td>0.014%</td>
<td>0.003%</td>
<td>2.005%</td>
</tr>
<tr>
<td>Average</td>
<td>0.158%</td>
<td>1.629%</td>
<td>0.228%</td>
<td>0.028%</td>
<td>1.136%</td>
</tr>
</tbody>
</table>

In terms of the number of start-ups and shutdowns, NSS, it can be observed that both strategies give the same results for nearly all simulations. The only difference between one start-up/shutdown is the CH simulation that considers \( \gamma_{\text{in}} \) as 25% of \( W_{\text{max}} \). Thus, this illustrates that the linearization errors from the concave approximation via CH can affect the number of GUs dispatched. Finally, as expected, NSS decreases as \( \gamma_{\text{in}} \) increases. Now, w.r.t. the computing times, the results show that PROP gives times slightly worse than CH and significantly better than MC. Considering that PROP captures the nonlinearities of the problem, this increase in computing time compared to CH can be considered negligible. Note that PROP took less than half a minute to solve a 5-plant, 18-GU cascade HUC problem. Finally, when analyzing the number of iterations, IT, it is noticed that the algorithm takes at most seven iterations to reach the stopping criterion considered. Fig. 6 presents the difference between OBJ in two successive iterations of the PROP algorithm. Note that the difference decreases exponentially at early iterations.

Figure 6. The difference in OBJ in successive iterations of PROP.

2) Comparing PROP with LACC

Table 8 shows the difference (DIFPROP) between PROP and LACC in OBJ and ST for \( \gamma_{\text{in}} = 0\% \). The LACC considers four different discretizations for the bivariate HPF. The 3Hx3Q, for example, contemplates three points equally distributed on the head and five on the turbined outflow. As a result, the HPF PWL is modeled by 16 triangles. Notice that the number of points on each axis in the LACC must attend \( 2^N \). Thus, limiting the simulation times, as illustrated in the table, was necessary to obtain low optimality gaps (GAP).

Table 8 – Difference between PROP and LACC

<table>
<thead>
<tr>
<th>HPF discretization</th>
<th>3Hx5Q</th>
<th>3Hx9Q</th>
<th>3Hx17Q</th>
<th>3Hx33Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangles</td>
<td>16</td>
<td>22</td>
<td>64</td>
<td>128</td>
</tr>
<tr>
<td>Time limit (s)</td>
<td>2,000</td>
<td>10,000</td>
<td>70,000</td>
<td>250,000</td>
</tr>
<tr>
<td>GAP</td>
<td>0.41%</td>
<td>0.15%</td>
<td>0.22%</td>
<td>0.21%</td>
</tr>
<tr>
<td>OBJ</td>
<td>91,946.7</td>
<td>91,473.8</td>
<td>91,235.2</td>
<td>91,155.1</td>
</tr>
<tr>
<td>DIFPROP</td>
<td>0.87%</td>
<td>0.36%</td>
<td>0.09%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

Note, in Table 8, that the greater the number of triangles in LACC, the closer the OBJ value is to that obtained via PROP. As the problems solved in PROP consider 55 points in the univariate HPFs in each iteration and the \( GHN \) is updated by the
nonlinear formulation, it can be considered that the PROP is more accurate than the LACC with finer discretization (3Hx33Q). However, while the more accurate LACC required 250,000 seconds (almost three days), PROP is solved in a time less than 30 seconds. In Figure 9, it is possible to observe that the \( GH_i \) obtained by PROP is similar to LACC in the finest discretization, which proves the effectiveness of the \( GH_i \) update along with the PROP iterations.

**B. Results for the cascade with four plants and 53 GUs**

The hydraulic configuration of the system with four plants can be seen in Fig. 7.

![Figure 7. Configuration of the cascade with four plants.](image)

Table 9 presents the main characteristics of the plants in this cascade with 53 GUs. Plant #2 has three groups of identical GUs, while the other plants have one single group with identical GUs. The water traveling time between any two coupled reservoirs is one hour.

Table 9 – Main operational characteristics of the plants of the system with 53 GUs.

<table>
<thead>
<tr>
<th>Plant</th>
<th>Capacity (MW)</th>
<th>( I_p )</th>
<th>( W_{	ext{output}} ) (m³/s)</th>
<th>( G_{	ext{output}} ) (MW)</th>
<th>( \rho_{	ext{output}} ) (hᵢ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>807.5</td>
<td>5</td>
<td>100/436</td>
<td>65.0/161.5</td>
<td>9,923/13,372</td>
</tr>
<tr>
<td>#2</td>
<td>3,440.0</td>
<td>11</td>
<td>120/480</td>
<td>70.0/176.0</td>
<td>25,467/34,432</td>
</tr>
<tr>
<td>#3</td>
<td>1,551.2</td>
<td>14</td>
<td>220/596</td>
<td>44.0/110.8</td>
<td>2,450/3,354</td>
</tr>
<tr>
<td>#4</td>
<td>1,540.0</td>
<td>14</td>
<td>200/636</td>
<td>44.0/110.0</td>
<td>14,400/20,000</td>
</tr>
</tbody>
</table>

Table 10 shows the coefficients of the net-head function for each plant. The coefficients \( A_{ij} \) of the efficiency functions are omitted due to confidentiality requirements.

![Figure 8. Generation targets for the four-plants cascade.](image)

The initial reservoir volumes and the incremental inflows are shown in Table 11.

Table 10 - Initial volumes and incremental inflows.

<table>
<thead>
<tr>
<th>Plant</th>
<th>( V_0 )</th>
<th>( \Delta V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>11,897</td>
<td>31,249</td>
</tr>
<tr>
<td>#2</td>
<td>450</td>
<td>2,110</td>
</tr>
<tr>
<td>#3</td>
<td>2,945</td>
<td>29,467</td>
</tr>
<tr>
<td>#4</td>
<td>9,923</td>
<td>14,573</td>
</tr>
</tbody>
</table>

The main results of PROP can be seen in Table 12.

Table 11 – Main results in system with 53 GUs

<table>
<thead>
<tr>
<th>Plant</th>
<th>OBJ</th>
<th>25%</th>
<th>50%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>498,582.7</td>
<td>503,860.9</td>
<td>506,212.9</td>
<td>508,556.2</td>
</tr>
<tr>
<td>#2</td>
<td>59</td>
<td>20</td>
<td>13</td>
<td>9</td>
</tr>
<tr>
<td>#3</td>
<td>47.91</td>
<td>43.41</td>
<td>52.04</td>
<td>46.77</td>
</tr>
<tr>
<td>#4</td>
<td>4</td>
<td>4</td>
<td>9</td>
<td>4</td>
</tr>
</tbody>
</table>

The results endorse the ability of PROP to solve problems with many GUs in a low computational time. Note that, in less than one minute, PROP can accurately solve (as it considers many points of the univariate nonlinear HPF and updates the gross head by the nonlinear formulation) the HUC problem in a cascade with four plants and 53 GUs of six different types.
This paper proposes an algorithm for solving the HUC by iteratively updating the gross head from approximate univariate HPFs via partitioning of the concave and convex regions. The proposed approach is tested in two cascade power plant systems. For the cascade with five plants and 18 GUs, the results show that the proposed algorithm, PROP, presents more accurate results when compared to algorithms that consider a concave approximation for each univariate HPF at a price of a mild computational performance. Compared to the algorithm considering the classical nonconvex approach MC, PROP gives identical results while significantly improving the computing times. Finally, the comparison of PROP with the MILP problem that considers a bivariate HPF via the LACC model shows that PROP, solved in less than 30 seconds, achieves results as good as the most accurate version of the LACC that takes about three days for finding the same solution. In the tests performed for a cascade with four plants and 53 GUs, the strategy PROP solves the problem in less than one minute for a fine discretization of each HPF. Therefore, the PROP can combine the precise representation of the nonconvexities of the HPFs with a good computational performance.

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