A Flexible Loss Reduction Formulation for Simultaneous Capacitor Placement and Network Reconfiguration in Distribution Grids

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***Abstract*—Efficiently reducing power losses involves various strategies such as opening normally closed switches and closing normally open tie-lines in distribution feeders, and optimally placing shunt capacitors in distribution networks. In this process, the accurate modeling of the demand side is crucial in understanding the behavior of electricity consumers appropriately. It is important to note that loads are not constant; they vary with changes in voltage magnitude, which in turn depend on the type of consumer. Each load can be represented by its constant power, current, and impedance components. In this regard, establishing a clear relationship between these components and consumer types can significantly enhance the flexibility of the network switching or capacitor placement strategy. Therefore, this study aims to mathematically formulate the correlation between load components and consumer types, aiming at establishing an efficient reconfiguration and capacitor allocation formulations. This is achieved by transforming polynomial load formulations into quadratic ones, while establishing mathematical relationships between the quadratic and polynomial models for various load types. The accuracy and convergence time of the proposed model was tested through its application to 16- and 33-bus distribution networks, and the results have been compared with the nonlinear metaheuristic approach. The results demonstrated that the proposed framework can efficiently provide optimal solutions within a shorter computational time as compared to the nonlinear and metaheuristic approaches.**

***Index Terms*--Capacitor placement, distribution feeders, flexible model, hybrid optimization.**

I. INTRODUCTION

When it comes to distribution feeders' reconfiguration and shunt capacitor placement for network loss minimization, the demand side plays a pivotal role [1]. This is because any alteration in load power directly influences the operational decisions of the system. Distribution feeders, which constitute a crucial component within power systems are responsible for delivering electric energy from generation centers to end-users through transmission lines [2]. Traditionally, modifying the configuration of distribution networks which is known as feeders' reconfiguration involves the manipulation of sectional branches and tie-lines of radial feeders, which aims at reducing the network losses [3]. Additionally, the optimal placement of capacitors is determined to minimize the active losses of the network. Shunt capacitors by compensating the part of reactive power demand of load causes a reduction in magnitude of feeders' current and subsequent power losses of distribution network [4]. It is worth noting that energy loss has a direct impact on operational costs and system efficiency, as well as indirectly affecting the quality of delivered power to consumers [5].

In recent times, there has been a growing reliance on reactive compensators to stabilize power systems under variable operational conditions. Reactive compensators are devices for injection or intake of reactive power in electric grids. Notably, the installation of shunt capacitors in reconfigurable distribution networks yields more significant loss reduction compared to their use in non-reconfigurable systems. This underscores the critical importance of simultaneously reconfiguring distribution feeders and optimally placing capacitors to ensure voltage stability of power system while curbing energy losses [6]. Numerous research papers have explored varied models for minimizing distribution losses by concurrently reconfiguring feeders and deploying reactive power compensators. However, a significant portion of these studies conducted feeder reconfigurations and capacitor allocations without considering load components and consumption types. Each load has three constant-power, constant-current, and constant-impedance components and different types such as industrial, commercial, and residential. In contrast, some research endeavors addressed these critical factors in their proposed formulations but did not explicitly calculate the relationship between load components and load types.

In [7], the problem of minimizing power losses through reconfiguration and capacitor allocation was addressed using simulated annealing (SA) across various scenarios. In the initial two scenarios, the reconfiguration and capacitor setting challenges were tackled separately. However, in the third scenario, capacitor banks were optimally placed in the distribution systems both before and after network reconfiguration. The fourth and final scenarios involved the simultaneous resolution of reconfiguration and capacitor allocation problems. The findings revealed that incorporating optimal capacitor allocation within the reconfiguration problem led to more efficient reductions in active power losses. SA is a well-established random search algorithm inspired by the physical process of annealing in solids. Nonetheless, the iterative execution of power flow calculations during the annealing process can be quite time-consuming when applied to the simultaneous reconfiguration and capacitor allocation of large distribution networks. Hence, in order to tackle the problem presented in [7], researchers of [8] turned to utilizing the ant colony optimization (ACO) algorithm. Their findings pointed to the fact that the simultaneous feeder switching and capacitor placement resulted in lower active power losses compared to solving the reconfiguration problem without the inclusion of shunt capacitor allocation. Their study also demonstrated that while ACO produced favorable results, including a reduction in losses and average CPU time compared to SA, metaheuristic approaches like ACO cannot guarantee optimal solutions.

In [9], a novel mathematical formulation was introduced to address network reconfiguration and capacitor placement while taking into account the costs associated with providing reactive power. This innovative methodology incorporated a sensitivity index to determine the optimal switch status for minimizing losses, considering different levels of the daily load curve. The approach followed a sequential process: first, the capacitor placement problem was tackled using a heuristic constructive algorithm (HCA), both before and after network reconfiguration. The results demonstrated that this approach required minimal computational effort and proved effective in medium-sized distribution networks with multiple load levels. However, it is important to note that the findings in [7] supported the idea that solving the capacitor placement problem either before or after network reconfiguration did not yield optimal solutions. Instead, both problems should be addressed simultaneously for more favorable outcomes.

In [10], researchers employed a specialized genetic algorithm (GA) to address the combined challenge of network reconfiguration and capacitor placement across three distinct load levels: light, average, and heavy. Their primary objective was to minimize both capacitor investment and power loss costs. The proposed GA approach began by constructing the initial population through a heuristic algorithm. This heuristic algorithm relied on two sensitivity indices, effectively reducing the search space and computational load. However, it is important to note that random search algorithms like GA do not provide a guarantee of achieving an optimal solution.

In [11], researchers tackled the intricate challenge of simultaneous network reconfiguration and capacitor placement in the presence of distributed generation (DG). Their approach considered various costs, including switching costs, expenses related to purchasing power from the substation, customer interruption costs, and transformer loss of life costs. They applied the ACO algorithm for this purpose. The results obtained from simulations indicated that the simultaneous coordination of feeder and capacitor switching could yield a greater reduction in the total grid cost compared to traditional reconfiguration or capacitor switching.

In [12], the reconfiguration and capacitor placement problem was simultaneously solved using a particle swarm optimization (PSO) algorithm to control the voltage rise and drop due to use of DG units in the distribution system. The objective was to minimize active power losses and voltage deviation in the presence of load and renewable generation uncertainty. Also, in [13], a mixed-integer linear programming (MILP) model was developed for power loss reduction via simultaneous reconfiguration and optimal capacitor placement in radial distribution networks using analytical methods.

In [14], the simultaneous optimal allocation of capacitor banks and distributed generators, as well as optimal radial distribution system reconfiguration was accomplished using a quasi-reflection-based slime mould algorithm (QRSMA). The objective was to reduce power losses and improve voltage stability and network reliability. Also, in [15], a new antlion optimizer (ALO) algorithm was proposed for reconfiguration and capacitor allocation in distribution networks with DG sources. The results indicate good performance of ALO for losses minimization and power quality improvement in radial systems. Moreover, in [16], a mixed-integer quadratic programming (MIQP) model was proposed to reconfigure distribution grids with switched capacitor banks using Benders decomposition in order to mitigate voltage volatility induced by renewable DG sources.

However, the models presented in [7]-[16] did not consider load components and consumption types in their analyses. Considering load type and components in distribution system operation is vital due to high dependency of different load types and components on voltage magnitude. Any change in voltage affects load power and therefore causes different consumption levels in the power grid. In this case, power losses change and new switching and capacitor placement strategies are needed to mitigate these incremental losses. To address this challenge, simultaneous network reconfiguration and capacitor placement in the presence of voltage-dependent loads was proposed in [17]. Instead of using the time-consuming Monte Carlo simulation (MCS), the reliability assessment was done using the Weibull-Markov stochastic model. This approach aimed to minimize interruption costs at the load point, including customer damage functions and their associated probabilities, all with the goal of enhancing the reliability level, reducing power losses, and minimizing the costs associated with capacitor installation. The simulation outcomes conclusively demonstrated that the deployment of shunt capacitors not only led to reductions in losses but also significantly improved network reliability in a cost-effective manner.

However, the model presented in [17], which is inherently nonlinear, was tackled using a gravitational search algorithm (GSA) without determining the load components' relationship with consumption type. Formulating this relationship is very important in operation and reconfiguration studies because it facilitates solving reconfiguration and capacitor placement problems even when load components data are unavailable. In addition, it is important to note that employing metaheuristic algorithms like GSA for computing nonlinear models does not provide a guarantee of obtaining optimal solutions. Additionally, utilizing classic optimization tools with nonlinear solvers to compute these models can be a time-consuming process. In contrast, the techniques applied for linearization should be carefully chosen to ensure that they do not significantly compromise the accuracy of the nonlinear models [18]. Hence, the present paper introduces a flexible and efficient framework for reconfiguration and capacitor allocation by introducing the relationship between load components and type. It achieves this by transforming polynomial load formulations into quadratic ones. To do this, the study establishes mathematical relationships between quadratic and polynomial models for various load types. The model flexibility means that the proposed approach can be used even when load component data are not available. Whereas inflexible models presented in existing research can only be utilized if load components characteristics are given. Unlike methods presented in [7], [10], and [17] which cannot provide optimality guarantees due to stochastic nature of metaheuristic algorithms such as SA [7] and GA [10], [17], the proposed model can be computed by analytical methods and commercial linear solvers. Analytical approaches and classic linear solvers converge to a single solution in all run efforts, while SA and GA suggest distinct solutions in their different runs. However, it should be mentioned that even though metaheuristics do not provide global optimality guarantees, they reach locally (near) optimal non-exact solutions. The simulation results highlight the superior efficiency and flexibility of the model formulated based on the correlation between load components and consumption types. Accordingly, the main novelties and contributions of the paper are:

- Presenting a linear formulation for the relationship between components of load and its type.
- Development of a model for simultaneous network reconfiguration and capacitor placement in distribution systems with voltage-dependent loads that can be easily computed by linear solvers compared to nonlinear models which are hardly solved by commercial optimization tools or may be computed by metaheuristics with uncertain solutions.
- Design of a generalized formulation for distribution system reconfiguration and capacitor allocation when data for load components are not available.

The remainder of the paper is organized as follows. Load components in terms of polynomial values are modeled in section II. In section III, the calculation of load components from polynomial values is explained. Also, simultaneous feeder reconfiguration and capacitor placement problems are formulated in section IV. In section V, the method of embedding the load model in the reconfiguration problem is described. Moreover, computational results are presented in section VI. Lastly, conclusions about the model and simulation results are discussed in section VII.

II. Representation of Load Components In Terms of Polynomial Values

In order to efficiently illustrate the relationship between load power and consumption type, the polynomial model developed by [19] can be used. This approach establishes the correlation between the specific power demand and its type through the following equations.

$$Pd_i / Pd_i^n = \sum_y A_y \left(V_i / V_n\right)^{\alpha_y} \tag{1}$$

$$Qd_i / Qd_i^n = \sum_y B_y \left(V_i / V_n\right)^{\beta_y} \tag{2}$$

Where, Pd_i, Qd_i, and V_i represent the real and reactive powers and voltage magnitude of load at node i, while Pd_i^n, Qd_i^n, and V_n denote their respective nominal values. Additionally, α_y and β_y are the polynomial exponents associated with load type y. The multipliers A_y and B_y account for the contributions of the real and reactive load at each node, respectively.

$$\sum_y A_y = \sum_y B_y = 1 \tag{3}$$

According to calculations done in Appendix A and utilizing (3), (1) and (2) can be rewritten as follows.

$$\begin{aligned} Pd_i / Pd_i^n = &\sum_y 0.5 A_y \alpha_y \left(\alpha_y - 1\right)\left(V_i / V_n\right)^2 + \\ &\sum_y A_y \alpha_y \left(2 - \alpha_y\right)\left(V_i / V_n\right) + \sum_y 0.5 A_y \alpha_y \left(\alpha_y - 3\right) + 1 \end{aligned} \tag{4}$$

$$\begin{aligned} Qd_i / Qd_i^n = &\sum_y 0.5 B_y \beta_y \left(\beta_y - 1\right)\left(V_i / V_n\right)^2 + \\ &\sum_y B_y \beta_y \left(2 - \beta_y\right)\left(V_i / V_n\right) + \sum_y B_y 0.5 \beta_y \left(\beta_y - 3\right) + 1 \end{aligned} \tag{5}$$

Conversely, power demand expressed in terms of load components can be represented by (6) and (7) [20].

$$Pd_i / Pd_i^n = \sum_z C_z \left(V_i / V_n\right)^z \tag{6}$$

$$Qd_i / Qd_i^n = \sum_z D_z \left(V_i / V_n\right)^z \tag{7}$$

Where, coefficients C_z and D_z represent percentages of real and reactive load component z at each node, respectively. It is essential that these coefficients satisfy (8).

$$\sum_z C_z = \sum_z D_z = 1 \tag{8}$$

Considering the existence of three components for each nodal load, namely, constant-power (z=0), constant-current (z=1), and constant-impedance (z=2), equations (6) and (7) can be expanded as follows.

$$Pd_i / Pd_i^n = C_2 \left(V_i / V_n\right)^2 + C_1 \left(V_i / V_n\right) + C_0 \tag{9}$$

$$Qd_i / Qd_i^n = D_2 \left(V_i / V_n\right)^2 + D_1 \left(V_i / V_n\right) + D_0 \tag{10}$$

$$C_0 + C_1 + C_2 = D_0 + D_1 + D_2 = 1 \tag{11}$$

When equation (9) is compared with (4) and equation (10) with (5), the relationship between the real and reactive load components and the consumption type is established as follows.

$$C_0 = 1 + 0.5 \sum_y A_y \alpha_y \left(\alpha_y - 3\right) \tag{12}$$

$$C_1 = \sum_y A_y \alpha_y \left(2 - \alpha_y\right) \tag{13}$$

$$C_2 = 0.5 \sum_y A_y \alpha_y \left(\alpha_y - 1\right) \tag{14}$$

$$D_0 = 1 + 0.5 \sum_y B_y \beta_y \left(\beta_y - 3\right) \tag{15}$$

$$D_1 = \sum_y B_y \beta_y \left(2 - \beta_y\right) \tag{16}$$

$$D_2 = 0.5 \sum_y B_y \beta_y \left(\beta_y - 1\right) \tag{17}$$

III. Determining Load Components Using Polynomial Values

Table I provides values of α_y and β_y for three distinct types of loads: industrial (y=1), residential (y=2), and commercial (y=3).

TABLE I. POLYNOMIAL VALUES

References	α_y			β_y		
	y=1	y=2	y=3	y=1	y=2	y=3
[20],[21]	0.18	0.92	1.51	6	4.04	3.4
[17],[22],[23]	0.1	1.7	0.6	0.6	2.6	2.5
[24]	0.18	1.04	1.5	6	4.19	3.15

Using equations (12) to (17), the constant-power, constant-current, and constant-impedance components of industrial, residential, and commercial loads based on the polynomial values presented in the first row of Table I are calculated as follows.

$$\begin{gathered} C_0 = 1 - 0.2538A_1 - 0.9568A_2 - 1.1249A_3 \\ C_1 = 0.3276A_1 + 0.9936A_2 + 0.7399A_3 \\ C_2 = -0.0738A_1 - 0.0368A_2 + 0.3851A_3 \\ D_0 = 1 + 9B_1 + 2.1008B_2 + 0.68B_3 \\ D_1 = -24B_1 - 8.2416B_2 - 4.76B_3 \\ D_2 = 15B_1 + 6.1408B_2 + 4.08B_3 \end{gathered} \tag{18}$$

When applying the identical calculation approach to the second and third rows of Table I, equations (19) and (20) yield.

$$\begin{gathered} C_0 = 1 - 0.145A_1 - 1.105A_2 - 0.72A_3 \\ C_1 = 0.19A_1 + 0.51A_2 + 0.84A_3 \\ C_2 = -0.045A_1 + 0.595A_2 - 0.12A_3 \\ D_0 = 1 - 0.72B_1 - 0.52B_2 - 0.625B_3 \\ D_2 = -0.12B_1 + 2.08B_2 + 1.875B_3 \\ D_1 = 0.84B_1 - 1.56B_2 - 1.25B_3 \end{gathered} \tag{19}$$

$$\begin{gathered} C_0 = 1 - 0.2538A_1 - 1.0192A_2 - 1.125A_3 \\ C_1 = 0.3276A_1 + 0.9984A_2 + 0.75A_3 \\ C_2 = -0.0738A_1 + 0.0208A_2 + 0.375A_3 \\ D_0 = 1 + 9B_1 + 2.4931B_2 + 0.2362B_3 \\ D_1 = -24B_1 - 9.1761B_2 - 3.6225B_3 \\ D_2 = 15B_1 + 6.6831B_2 + 3.3863B_3 \end{gathered} \tag{20}$$

Tables II, III, and IV present the results of equations (18) to (20) for various load types. The load components presented in these tables were computed by substituting specific values for the coefficients. For industrial loads, A_1=B_1=1 and A_2=A_3=B_2=B_3=0; for residential loads, A_2=B_2=1 and A_1=A_3=B_1=B_3=0; and for commercial loads, A_3=B_3=1 and A_1=A_2=B_1=B_2=0 are used in equations (18) to (20). In Table II, A_1=A_2=A_3=B_1=B_2=B_3=1/3 is assigned. In Table III, the coefficients were set as A_1=A_2=B_1=B_2=1/4 and A_3=B_3=1/2. Lastly, in Table IV, A_1=B_1=1/2, A_2=B_2=1/8 and A_3=B_3=3/8 are considered for mixed loads.

TABLE II. FINDINGS FROM THE COMPUTATION OF LOAD COMPONENTS BASED ON DATA FROM REFERENCES [20] AND [21]

Load types	C_z			D_z		
	z=0	z=1	z=2	z=0	z=1	z=2
Ind.	0.7462	0.3276	-0.0738	10	-24	15
Res.	0.0432	0.9936	-0.0368	3.1008	-8.2416	6.1408
Com.	-0.1249	0.7399	0.3851	1.68	-4.76	4.08
Mix.	0.2215	0.6870	0.0915	4.927	-12.334	8.407

TABLE III. FINDINGS FROM THE COMPUTATION OF LOAD COMPONENTS BASED ON DATA FROM REFERENCES [17], [22], AND [23]

Load types	C_z			D_z		
	z=0	z=1	z=2	z=0	z=1	z=2
Ind.	0.855	0.19	-0.045	0.28	0.84	-0.12
Res.	-0.105	0.51	0.595	0.48	-1.56	2.08
Com.	0.28	0.84	-0.12	0.375	-1.25	1.875
Mix.	0.3275	0.595	0.0775	0.3775	-0.805	1.4275

TABLE IV. FINDINGS FROM THE COMPUTATION OF LOAD COMPONENTS BASED ON DATA FROM REFERENCE [24]

Load types	C_z			D_z		
	z=0	z=1	z=2	z=0	z=1	z=2
Ind.	0.7462	0.3276	-0.0738	10	-24	15
Res.	-0.0192	0.9984	0.0208	3.493	-9.1761	6.6831
Com.	-0.125	0.75	0.375	1.2362	-3.6225	3.3863
Mix.	0.3238	0.5698	0.1063	5.9	-14.505	9.6052

It is worth noting that the multipliers A_y are constrained to be positive real numbers falling within the range of 0 to 1, and their total sum must equal one. On the other hand, coefficients C_z are real numbers, which can be either negative or positive and may vary in magnitude, potentially exceeding 1. It is essential to emphasize that the sum of C_z coefficients for a specific load type should equal one. This aspect is evident in the tables mentioned above, as the summation of numbers in each row, whether for real or reactive loads, always amounts are equal to one. Equations (12) to (17) serve as valuable relationships for calculating load components across various load types.

IV. Modeling the Simultaneous Reconfiguration of Networks and Capacitor Placement

Simultaneous reconfiguration of radial feeders and capacitor placement in distribution networks can be formulated as a power loss optimization problem using (21) to (35).

$$\text{Min} \sum_{ij \in \psi^l} r_{ij} I_{ij}^{\;2} \tag{21}$$

s.t.:

$$Ps_i + \sum_{ki \in \psi^l} P_{ki} - \sum_{ij \in \psi^l} P_{ij} - \sum_{ij \in \psi^l} r_{ij} I_{ij}^{\;2} = Pd_i \quad \forall i \in \psi^b \tag{22}$$

$$Qs_i + \sum_{ki \in \psi^l} Q_{ki} - \sum_{ij \in \psi^l} Q_{ij} - \sum_{ij \in \psi^l} x_{ij} I_{ij}^{\;2} = Qd_i - Qc_i \quad \forall i \in \psi^b \tag{23}$$

$$V_i^{\;2} - V_j^{\;2} = 2\left[r_{ij}P_{ij} + x_{ij}Q_{ij}\right] + \left(r_{ij}^{\;2} + x_{ij}^{\;2}\right) I_{ij}^{\;2} + b_{ij} \quad \forall i \neq j \in \psi^b, ij \in \psi^l \tag{24}$$

$$V_j^{\;2} I_{ij}^{\;2} = P_{ij}^{\;2} + Q_{ij}^{\;2} \quad \forall ij \in \psi^l \tag{25}$$

$$V_l^2 \le V_i^2 \le V_u^2 \quad \forall i \in \psi^b \tag{26}$$

$$0 \le I_{ij}^{\;2} \le Iu_{ij}^{\;2} y_{ij} \quad \forall ij \in \psi^l \tag{27}$$

$$\left|P_{ij}\right| \leq V_u Iu_{ij} y_{ij} \quad \forall ij \in \psi^l \tag{28}$$

$$\left|Q_{ij}\right| \leq V_u Iu_{ij} y_{ij} \quad \forall ij \in \psi^l \tag{29}$$

$$\left|b_{ij}\right| \leq \left(V_u^2 - V_l^2\right)\left(1 - y_{ij}\right) \quad \forall ij \in \psi^l \tag{30}$$

$$\sum_{ij \in \psi^l} y_{ij} = \left|\psi^b\right| - 1 \tag{31}$$

$$Qc_i = \sum_{b \in \psi^c} x_{i,b} Q_b \qquad \forall i \in \psi^d \tag{32}$$

$$\sum_{b \in \psi^c} x_{i,b} \leq 1 \qquad \forall i \in \psi^d \tag{33}$$

$$0 \leq Qs_i \leq Ps_i \tan(\cos^{-1}(pf)) \qquad \forall i \in \psi^s \tag{34}$$

$$\sum_{b \in \psi^c} \sum_{i \in \psi^d} x_{i,b} Q_b \leq \sum_{i \in \psi^d} Qd_i \tag{35}$$

In the aforementioned equations, the symbols ψ^l, ψ^b, ψ^s, ψ^d, and ψ^c respectively represent sets of distribution lines, network buses, substation nodes, load points, and capacitor banks. r_{ij} is electric resistance, and x_{ij} is the ohmic reactance of line ij. Ps_i, Qs_i, Qc_i, V_l, and V_u are substation real and reactive powers, capacitor reactive injection, and lower and upper voltage limits, respectively. y_{ij} and $x_{i,b}$ are binary variables for indicating the switching state of line ij and the connection of capacitor bank b to bus i, respectively (0 for open switches and not connected capacitors and 1 for closed switches and connected capacitor banks). P_{ij}, Q_{ij}, I_{ij}, and Iu_{ij} are real and reactive powers, current magnitude, and the upper current limit of line ij, respectively. The variable b_{ij} serves to establish KVL in loops formed by line ij (b_{ij}=0) and is ignored for disconnected lines ($b_{ij}\neq$0). Lastly, Q_b is the size of a capacitor bank and pf is power factor of substations.

Equations (22) and (23) address the real and reactive power balances at bus i, respectively. Equation (24) represents KVL applied to all planar loops within the distribution system. It means that if switch of line ij be open, KVL is not required to be satisfied for the loop related to this line, because I_{ij}, P_{ij}, and Q_{ij} will be zero in (24) and amount of variable b_{ij} will equal to a real number according to (30). On the other hand, if the switch of line ij is closed, b_{ij} will be zero due to y_{ij}=1 in (30) and therefore voltage drop of the line ij will be equal to right side of equation (24) with b_{ij}=0, i.e. KVL is satisfied for the loop consists of line ij. Equation (25) describes the relationship between line power and its real and reactive components. This means that square of absolute amount for complex power in line ij, i.e. $V_j^2 I_{ij}^2$ will be equal to square of its real (P_{ij}) and imaginary (Q_{ij}) parts. Constraints (26) to (30) define the permissible limits for bus voltage, line current, real and reactive power flows, and variable b_{ij}, respectively. Equation (31) enforces the radial operation of the distribution system. Lastly, (32) to (34) impose constraints on reactive power generation, the number of shunt capacitors, and reactive power supply at substation buses. Equation (35) states that the total reactive power injection from shunt capacitors must be less than the total reactive power consumption. However, it is important to note that the relationships presented in equations (21) to (35) constitute a complex nonlinear optimization problem, demanding significant computational efforts and processing time. To address this challenge, the following model is proposed by substituting nonlinear terms of I_{ij}^2, V_i^2 and V_j^2 in (21) to (27) with linear terms of I_{ij}^s, V_i^s and V_j^s, respectively, and replacing (31) by set of equations (43) to (46). This model can be simply computed by linear solvers in any commercial software.

$$\text{Min} \sum_{ij \in \psi^l} r_{ij} I_{ij}^s \tag{36}$$

s.t. (28)–(30), (32)–(35), and the following constraints:

$$Ps_i + \sum_{ki \in \psi^l} P_{ki} - \sum_{ij \in \psi^l} P_{ij} - \sum_{ij \in \psi^l} r_{ij} I_{ij}^s = Pd_i \quad \forall i \in \psi^b \tag{37}$$

$$Qs_i + \sum_{ki \in \psi^l} Q_{ki} - \sum_{ij \in \psi^l} Q_{ij} - \sum_{ij \in \psi^l} x_{ij} I_{ij}^s = Qd_i - Qc_i \quad \forall i \in \psi^b \tag{38}$$

$$V_i^s - V_j^s = 2\left[r_{ij} P_{ij} + x_{ij} Q_{ij}\right] + \left(r_{ij}^2 + x_{ij}^2\right) I_{ij}^s + b_{ij} \quad \forall i \neq j \in \psi^b, ij \in \psi^l \tag{39}$$

$$V_j^s I_{ij}^s \geq P_{ij}^2 + Q_{ij}^2 \;\; \forall ij \in \psi^l \tag{40}$$

$$V_l^2 \leq V_i^s \leq V_u^2 \quad \forall i \in \psi^b \tag{41}$$

$$0 \leq I_{ij}^s \leq Iu_{ij}^2 y_{ij} \;\; \forall ij \in \psi^l \tag{42}$$

$$y_{ij} = w_{ij} + w_{ji} \quad \forall ij \in \psi^l \tag{43}$$

$$\sum_{ij \in \psi^l} w_{ij} = 1 \tag{44}$$

$$w_{ij} = 0 \quad \forall i \in \psi^s, ij \in \psi^l \tag{45}$$

$$w_{ji} = 0 \quad \forall j \in \psi^s, ij \in \Omega^l \tag{46}$$

Here, w_{ij} represents the power flow direction of line ij, taking values of 0 or 1, and I_{ij}^s and V_i^s denote the squares of the variables I_{ij} and V_i, respectively. Equations (43) to (46) constitute a set of equations utilized to satisfy the radiality constraint, as they offer superior performance in large distribution systems when compared to equation (31).

V. Incorporating Load Modeling into the Simultaneous Network Reconfiguration and Capacitor Placement Problem

Given the challenges associated with solving equations (1) and (2) using commercial solvers, especially when compared to the ease of handling quadratic equations (9) and (10), along with the potential to calculate load components based on consumer type, the optimal solution strategy involves transforming the highly nonlinear polynomial framework into a second-order model utilizing equations (9) to (17). From Appendix B, by inserting equation (67) into equation (66) and subsequently inserting equation (66) into equations (64) and (65), (47) and (48) are obtained.

$$Pd_i / Pd_i^n = \left(0.5C_1 + C_2\right) V_i^s + C_0 + 0.5C \tag{47}$$

$$Pd_i / Pd_i^n = \left(0.5D_1 + D_2\right) V_i^s + D_0 + 0.5D_1 \tag{48}$$

Equations (47) and (48) align perfectly with the reconfiguration and capacitor placement problem outlined in equations (28) to (30) and (32) to (46).

VI. Computational Results

Expressions (28) to (30) and (32) to (46), along with equations (47) and (48), formulate a conic convex optimization problem that can be effectively solved using commercial linear solvers, because all terms of the model (28)–(30) and (32)–(48) are linear except for constraint (40). Expression (40) is a convex quadratic inequality constraint that can be easily computed by linear solvers. Hence, the proposed model was applied to two standard distribution systems using CPLEX within the AMPL software, considering various load types and components. It is important to mention that all computations were conducted on a processor with CPU of 3.6 GHz and 8 GB of RAM.

It is assumed that load components for simultaneous network reconfiguration and capacitor placement are available. Therefore, the same load combination of [17] with load components of C_0=0.337, C_1=0.546, C_2=0.117, D_0=0.38, D_1= –0.716, and D_2=1.338 were adopted for case 1. Also, load combination of [22] with components of C_0=0.2697, C_1=0.5455, C_2=0.1847, D_0=0.3933, D_1=–0.8515, and D_2=1.4583 were considered for case 2 and the load combination of [21] with C_0=0.2215, C_1=0.6870, C_2=0.0915, D_0=4.927, D_1= –12.334, and D_2=8.407 were set for case 3. The efficiency of the proposed reconfiguration and capacitor placement model was compared with those of the exact model and [17] for 16- and 33-bus distribution systems. It should be noted that data of these standard networks are available in [25]. The exact solutions were calculated by solving exponential load equations (1) and (2) and the coordinated model (21)–(35) using the nonlinear solver of KNITRO in AMPL. It is assumed that all load buses can be connected to capacitor banks of 150-kVAr [26]. The lower and upper voltage ranges are set to 0.9 pu and 1.1 pu [27]. The power factor of all substations is considered to be 0.8 [28]. Tables V to VIII present the computation results.

TABLE V. RESULTS OF EXACT AND FLEXIBLE MODELS FOR CASE 1 IN 33-BUS SYSTSEM

Results	Flexible Model	Nonlinear Model (Exact)	Model [17]
Optimal buses for capacitor installation	2,3,5,7,10,13, 15,17,23,32,33	2,3,5,7,10,13, 15,17,23,32,33	2,4,9,16,17, 19,23,27,32,33
Open switches	7,9,14,28,32	7,9,14,28,32	9,14,28,32,33
Total number of capacitors	11	11	11
Power losses (kW)	126.31	126.37	135.35
CPU time (s)	1.22	1676.18	9.43

TABLE VI. RESULTS OF EXACT AND FLEXIBLE MODELS FOR CASE 1 IN 16-BUS SYSTSEM

Results	Flexible Model	Nonlinear Model (Exact)
Optimal buses for capacitor installation	4,5,7,8,10,13, 15,16	4,7,9,10,12,13, 15,16
Open switches	18,20,25	18,20,25
Total number of capacitors	8	8
Power losses (kW)	56.24	56.32
CPU time (s)	2.84	1909.07

Tables V to VIII clearly demonstrate that the flexible switching and capacitor placement method outperforms both the nonlinear model and the model [17]. This superiority arises from the proposed coordinated reconfiguration and reactive power control approach, which not only achieves precise solutions similar to those obtained by KNITRO but also does so in significantly less computational time compared to the nonlinear method. Furthermore, it yields better solutions than the model presented in [17] because of lower power losses obtained in Table V.

TABLE VII. RESULTS OF EXACT AND FLEXIBLE MODELS FOR CASE 2 IN 16-BUS SYSTSEM

Results	Flexible Model	Nonlinear Model (Exact)
Optimal buses for capacitor installation	4,7,8,10,15	4,7,8,10,15
Open switches	18,20,25	18,20,25
Total number of capacitors	5	5
Power losses (kW)	57.27	57.28
CPU time (s)	0.91	7053.36

TABLE VIII. RESULTS OF EXACT AND FLEXIBLE MODELS FOR CASE 3 IN 16-BUS SYSTSEM

Results	Flexible Model	Nonlinear Model (Exact)
Optimal buses for capacitor installation	4,7,8,10,15	4,7,8,10,12
Open switches	18,20,25	18,20,25
Total number of capacitors	5	5
Power losses (kW)	57.27	57.37
CPU time (s)	0.84	3108.89

The flexible coordinated approach achieves nearly identical solutions as those calculated by the exact model for exponential values in [17], [21], and [22]. Remarkably, it does so at a significantly faster rate than KNITRO. This characteristic positions the method proposed in current research as a valuable model for rapid optimization tasks, especially within extensive distribution systems. Consequently, it can be inferred that utilizing the proposed network reconfiguration and capacitor placement strategy leads to optimal solutions, which are computed more efficiently than those obtained through the nonlinear model described in equations (1) and (2). These observations underscore the significance of representing load types in terms of load components.

While there may be a slight variation in losses compared to the exact model presented by the flexible formulation, it is worth noting that the nonlinear model is extremely time-consuming when it comes to network switching and allocating reactive power compensators in distribution systems. On the other hand, the flexible model efficiently approaches optimal solutions within a much shorter time frame than the nonlinear model. Despite achieving the same level of accuracy as the nonlinear model for coordinated network switching and reactive power controls, the model proposed, which relies on the correlation between load components and consumption type, exhibits greater computational efficiency compared to the exact model. This efficiency is particularly significant when considering the computational time required. The CPU time holds paramount importance in online operational applications, particularly in scenarios where line switching may undergo frequent changes at intervals such as every quarter, half, or hour. Furthermore, the solutions derived from the flexible model showcase a remarkable level of precision regarding the linearization technique detailed in Section V. This affirms that the model introduced in this current research serves as a suitable alternative to conventional network switching and shunt capacitor placement models. The straightforward implementation of the proposed formulation, utilizing linear solvers, coupled with its capacity to handle voltage fluctuations, positions it as a promising model with practical applications.

VII. Conclusion

Traditionally, the optimization of distribution system power losses through feeder reconfiguration and the optimal placement of reactive power compensators involves the operation of sectional and tie-line switches and the optimal installation of shunt capacitors, all tailored to a specific load demand. However, the outcome of these operations is heavily influenced by voltage fluctuations that arise after feeder reconfiguration. Voltage fluctuations, in turn, impact power demand due to the voltage dependency of load components and their respective types. This factor can introduce variability into the results of reconfiguration and capacitor placement. To address this important issue, the present research introduces an efficient and a flexible model that leverages the correlation between load components and consumer types. This model is formulated as mixed-integer conic optimization problems using CPLEX within the AMPL framework that is simply computed by linear solvers in contrast to nonlinear models. The proposed model provides accurate solutions for simultaneous feeder reconfiguration and capacitor placement and is well-suited for online applications due to its expedited computation. The analysis of the results reveals that the proposed model does not only reduce power losses but also accomplishes this in a significantly shorter computational time compared to the exact nonlinear model. Beyond its precision and ability to minimize power losses, the principal strength of the proposed formulation lies in its remarkably low computation time. Additionally, the strategy suggested by the paper offers a flexible framework for simultaneous feeder reconfiguration and capacitor allocation without relying on predefined load components. In this scenario, load components can be calculated from polynomial values and subsequently integrated into the flexible model. Furthermore, the formulation of the proposed model is straightforward, allowing for computation using any commercial linear solvers. It should be noted that the feeder reconfiguration and capacitor placement can be also used for enhancement of voltage stability. In addition, the proposed model may be applied to larger distribution systems. Therefore, the inclusion of bus voltage deviation in the objective function and testing the model on larger networks are future research lines of the present paper.

Appendix A

By adding "1" to equations (1) and (2) and subsequently subtracting "1" from these equations, (49) and (50) are obtained:

$$Pd_i / Pd_i^n = \sum_y A_y \left(1 - 1 + \left(V_i / V_n\right)\right)^{\alpha_y} \tag{49}$$

$$Qd_i / Qd_i^n = \sum_y B_y \left(1 - 1 + \left(V_i / V_n\right)\right)^{\beta_y} \tag{50}$$

Generally, due to $\left| V_i / V_n - 1 \right| < 1$, the binomial expansion of (49) and (50) is typically represented by (51) [29].

$$\begin{aligned} &\left(1 + \left(V_i/V_n\right) - 1\right)^x = 1 + x\left(\left(V_i/V_n\right) - 1\right) + 0.5x\left(x-1\right) \times \\ &\left(\left(V_i/V_n\right) - 1\right)^2 + \left(1/6\right)x\left(x-1\right)\left(x-2\right)\left(\left(V_i/V_n\right) - 1\right)^3 + \ldots \end{aligned} \tag{51}$$

In the above equation, x can be α_y or β_y. Due to network stability and security concerns, the ratio of V_i to V_n in power systems is close to unity, resulting in $\left| V_i / V_n - 1 \right| << 1$. Consequently, (51) can be efficiently approximated by (52) with a high degree of precision [30].

$$\left(1 + \left(V_i/V_n\right) - 1\right)^x \cong 1 + x\left(\left(V_i/V_n\right) - 1\right) + 0.5x\left(x-1\right)\left(\left(V_i/V_n\right) - 1\right)^2 \tag{52}$$

Substituting $x = \alpha_y$ and $x = \beta_y$ into (52) and then inserting (52) into (49) and (50) respectively, yields

$$Pd_i / Pd_i^n = \sum_y A_y \left(1 + \alpha_y \left(\left(V_i/V_n\right) - 1\right) + 0.5\alpha_y\left(\alpha_y - 1\right)\left(\left(V_i/V_n\right) - 1\right)^2\right) \tag{53}$$

$$Pd_i / Pd_i^n = \sum_y B_y \left(1 + \beta_y \left(V_i/V_n - 1\right) + 0.5\beta_y\left(\beta_y - 1\right)\left(\left(V_i/V_n\right) - 1\right)^2\right) \tag{54}$$

Expanding (53) and (54) results in the following equations.

$$\begin{aligned} &Pd_i / Pd_i^n = \sum_y 0.5 A_y \alpha_y \left(\alpha_y - 1\right)\left(V_i/V_n\right)^2 + \\ &\sum_y A_y \alpha_y \left(2 - \alpha_y\right)\left(V_i/V_n\right) + \sum_y 0.5 A_y \alpha_y \left(\alpha_y - 3\right) + \sum_y A_y \end{aligned} \tag{55}$$

$$\begin{aligned} &Qd_i / Qd_i^n = \sum_y 0.5 B_y \beta_y \left(\beta_y - 1\right)\left(V_i/V_n\right)^2 + \\ &\sum_y B_y \beta_y \left(2 - \beta_y\right)\left(V_i/V_n\right) + \sum_y B_y 0.5 \beta_y \left(\beta_y - 3\right) + \sum_y B_y \end{aligned} \tag{56}$$

From (11) and (12) to (17), it is necessary for the summations of the following terms to be equal to zero.

$$\sum_y \frac{1}{2} A_y \alpha_y \left(\alpha_y - 3\right) + \sum_y A_y \alpha_y \left(2 - \alpha_y\right) + \sum_y \frac{1}{2} A_y \alpha_y \left(\alpha_y - 1\right) = 0 \tag{57}$$

$$\sum_y \frac{1}{2} B_y \beta_y \left(\beta_y - 3\right) + \sum_y B_y \beta_y \left(2 - \beta_y\right) + \sum_y \frac{1}{2} B_y \beta_y \left(\beta_y - 1\right) = 0 \tag{58}$$

By factoring A_y in (57) and (58), the following equations are obtained:

$$\sum_y A_y \left(0.5\alpha_y^{\ 2} - 1.5\alpha_y + 2\alpha_y - \alpha_y^{\ 2} + 0.5\alpha_y^{\ 2} - 0.5\alpha_y\right) = 0 \tag{59}$$

$$\sum_y B_y \left(0.5\beta_y^{\ 2} - 1.5\beta_y + 2\beta_y - \beta_y^{\ 2} + 0.5\beta_y^{\ 2} - 0.5\beta_y\right) = 0 \tag{60}$$

Consequently:

$$\sum_y A_y \left(\alpha_y^{\ 2} - \alpha_y^{\ 2} + 2\alpha_y - 2\alpha_y\right) = 0 \tag{61}$$

$$\sum_y B_y \left(\beta_y^{\ 2} - \beta_y^{\ 2} + 2\beta_y - \beta\alpha_y\right) = 0 \tag{62}$$

Thus, all conditions are met, and equations (12) to (17) are mathematically valid.

APPENDIX B

The focus here is on solving equations (9) and (10) by representing them in the per unit (pu) system as follows.

$$\begin{aligned} Pd_i / Pd_i^n &= C_2 V_i^2 + C_1 V_i + C_0 \\ Qd_i / Qd_i^n &= D_2 V_i^2 + D_1 V_i + D_0 \end{aligned} \tag{63}$$

Replacing $V_i^2 = V_i^s$ in (63), raises

$$Pd_i / Pd_i^n = C_2 V_i^s + C_1 \left(V_i^s\right)^{0.5} + C_0 \tag{64}$$

$$Pd_i / Pd_i^n = D_2 V_i^s + D_1 \left(V_i^s\right)^{0.5} + D_0 \tag{65}$$

The second terms of (64) and (65) can be expressed as equation (66).

$$\left(V_i^s\right)^{0.5} = \left(1 - \left(1 - V_i^s\right)\right)^{0.5} \tag{66}$$

In the context of distribution system operation, when 0.5× $(1-V_i^s)$ is significantly less than 1, equation (66) can be effectively approximated as the linear equation (67).

$$\left(1 - \left(1 - V_i^s\right)\right)^{0.5} \cong 1 - 0.5\left(1 - V_i^s\right) = 0.5 V_i^s + 0.5 \tag{67}$$

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